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Dynamics of Infinitely Many Particles Mutually Interacting in Three Dimensions via a Bounded Superstable Long-Range Potential

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We show existence and uniqueness for the solutions to the Newton equations relative to a system of infinitely many particles moving in the three-dimensional space and mutually interacting via a bounded superstable long-range potential. The present paper complete an analogous result obtained for positive short-range interaction.

KEY WORDS: Infinite dynamics; superstable interaction; long-range potential.

1. INTRODUCTION

In the rigorous study of Non-Equilibrium Statistical Mechanics a first problem that arises is to give a precise sense to the time evolution of states of infinitely extended systems. In this paper we consider a physical system composed by infinitely many particles mutually interacting in three dimensions via a bounded superstable long-range potential. We want to establish existence and uniqueness of the time evolution of the system governed by Eq. (1.1), which means essentially to show that a quasi-local observable evolves remaining quasi-local. This paper extends the results of a previous paper by Caglioti *et al.*,[(2)] who consider particles interacting by means of a positive, bounded, finite-range potential. As it was claimed in ref.2, the extension to superstable potentials seems quite natural and the problems of such a generalization are essentially technical in nature.

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With the generalizations introduced in the present paper the more important potentials which are not yet included in this kind of analysis are those singular at the origin; they, although interesting from a physical point of view, seem to be out of a possible approach with the present techniques in three dimensions. An interaction which is singular at the origin in fact could produce a too fast growth of the maximal velocity assumed by the particles, which could diverge in a finite time. To confirm the difficulties that appear in three dimensions, Fritz and Dobrushin[(4)] have exhibited an example of a system of infinite particles with a hard-core potential, which preserves energy in the collisions but it's not hamiltonian, and it produces a collapse in three-dimensions but not in two. In one dimension the first pioneer papers on this subject go back to Lanford[(6,7)] who considered the case of bounded, short-range potentials, while the case of singular interactions was first treated in ref. 3. In two dimensions Fritz and Dobrushin solved the problem for finite-range potentials[(4)], whereas Fritz[(5)] extended the previous results in two dimensions for superstable, singular, finite-range potentials. The extension of this result for long-range potentials is due to Bahn *et al.*[(1)]

Let us briefly describe the contents of the present paper. We are going to consider the motion of a countable collection of identical particles of unit mass in the three-dimensional Euclidean space $\mathbb{R}^3$. A configuration of the system is represented as an infinite sequence $\{q_i, v_i\}_{i\in\mathbb{N}}$ of the positions and velocities of the particles, and its time evolution is characterized by the solutions of the Newton equations:

$$\ddot{q}_i(t) = \sum_{j\in\mathbb{N},\, j\neq i} F(q_i(t) - q_j(t)), \quad i \in \mathbb{N}, \tag{1.1}$$

where $F(x) = -\nabla\phi(x)$. We assume that ϕ is a symmetric pair potential, superstable, bounded, and of an infinite range, with a power-like decreasing rate (see Section 2 for the details).

The first mathematical problem that arises is to establish existence and uniqueness of the solutions of Eq. (1.1), which have to be complemented by the initial conditions $\{q_i(0), v_i(0)\}_{i\in\mathbb{N}}$. We can exhibit initial conditions that after a finite time produce a collapse of the system (i.e. infinitely many particles in a bounded region), so we must choose them in order to exclude these bad initial data, but taking into account all the relevant states from a thermodynamical point of view.

We organize this paper as follows. In Section 2, we describe the class of the interactions studied and we give the main results of the paper. In Section 3, we give an a priori estimate of the energy of a region of the space, whereas in Section 4, we give some dynamical estimates on the maximal velocity assumed by a single particle and on the work made by the system on it. Finally, thanks to these preliminary results, in Section 5, we prove the existence of the solutions of the Newton equations.

The Appendices are devoted to the proof of some technical results.

2. NOTATIONS, DEFINITIONS AND MAIN RESULTS

In this paper we study the dynamics of infinite particles moving in the Euclidean space $\mathbb{R}^3$. Let $X=\{q_i, v_i\}_{i\in\mathbb{N}}$ be the infinite sequence of positions and velocities of the particles. We assume that X is a locally finite configuration, that is in any compact set $\Lambda\subset\mathbb{R}^3$ the number of the particles in the region Λ:

$$n_\Lambda=\sum_{i\in\mathbb{N}}\chi(q_i\in\Lambda) \tag{2.1}$$

is finite. We denote by $\chi(A)$ the characteristic function of the set A, and by $B(\mu, R)$ the open ball centered in μ and of radius R. The integer part of the real number x is here denoted by $[x]$.

For simplicity in the sequel we will denote by $D_i, E_i, L_i, \widetilde{D}_i, \widetilde{E}_i, \widetilde{L}_i$ any positive constant, possibly depending on the interaction ϕ and on the initial configuration X of the system.

Let us now define the class of superstable interactions, which we are going to consider in this paper. Given a symmetric pair potential $\phi(x)\equiv\phi(|x|)$, $x\in\mathbb{R}^3$, continuous with its first and second derivatives, we give the following definition:

Definition 2.1 (Superstability). Let us divide the space $\mathbb{R}^3$ into cubes Δ_α of side 1 and centered in $\alpha\in\mathbb{Z}^3$. Let n_{Δ_α} be the number of particles in Δ_α.

We say that the potential ϕ is superstable if there exist constants $A>0$, $B\geqslant 0$ for which $\forall n$ and $\forall q_1,\dots,q_n$ we have:

$$U(q_1,\dots,q_n)\geqslant -Bn+A\sum_\alpha n^2_{\Delta_\alpha}, \tag{2.2}$$

with

$$U(q_1,\dots,q_n)=\frac{1}{2}\sum_{i\neq j}\phi(|q_i-q_j|).$$

A superstable potential can be decomposed into the sum of a stable potential plus a potential not negative, strictly positive at the origin.[8,9] In spite of the presence of an attractive part, superstability avoids large concentrations of particles in small regions of space.

Here we consider the interaction due to a superstable, bounded, long-range potential, with a power-like decreasing rate, for which there exist positive constants γ, G_1, G_2, G_3, r_0, such that, for $|x| > r_0$:

$$|\phi(x)| \leqslant \frac{G_1}{|x|^\gamma}, \tag{2.3}$$

$$|\nabla\phi(x)| \leqslant \frac{G_2}{|x|^{\gamma+1}} \tag{2.4}$$

and

$$|\nabla\phi(x) - \nabla\phi(y)| \leqslant \frac{G_3}{(1+\min(|x|,|y|))^{\gamma+2}}\,|x-y|. \tag{2.5}$$

In the sequel we assume $\gamma > 7$. This technical assumption will be discussed at the end of this section.

In order to consider configurations which are typical from a thermodynamical point of view, we must allow initial data with logarithmic divergences in the velocities and in the local densities.

More precisely, we define, using the short-hand notation $\phi_{i,j} = \phi(|q_i - q_j|)$,

$$Q(X;\mu,R) = \sum_{i\in\mathbb{N}} \chi(|q_i-\mu| \leqslant R)\left(\frac{v_i^2}{2} + \frac{1}{2}\sum_{\substack{j:j\neq i,\\ q_j\in B(\mu,R)}} \phi_{i,j} + b\right), \quad b > B \tag{2.6}$$

and

$$Q_\xi(X) = \sup_\mu \sup_{R:R>\psi_\xi(|\mu|)} \frac{Q(X;\mu,R)}{R^3}, \tag{2.7}$$

where

$$\psi_\xi(x) = \{\log(\max(x,e))\}^\xi, \quad x \in \mathbb{R}^+. \tag{2.8}$$

For each $\xi \geqslant 1/3$, the set of all configurations for which $Q_\xi(X) < \infty$ constitutes a full measure set for all Gibbs states associated to the particle system (see refs. 3 and 4).

If the initial configuration $X = \{q_i(0), v_i(0)\} \in \mathcal{X}_\xi$, with $\mathcal{X}_\xi = \{X : Q_\xi(X) < \infty\}$, we will make sense of the infinite set of Newton equations:

$$\ddot{q}_i(t) = F_i(X(t)) = \sum_{j \neq i} F_{i,j}(t), \tag{2.9}$$

where $F_{i,j} = -\nabla\phi(|q_i - q_j|)$ is the force exerted by the particle j on the particle i.

The solutions to the Newton equations will be constructed by means of a limiting procedure. Neglecting all the particles outside $B(0,n)$, we consider, for an integer n:

$$\begin{aligned} \ddot{q}_i^n(t) &= F_i^n(t), \\ q_i^n(0) &= q_i, \qquad v_i^n(0) = v_i, \quad i \in I_n, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} I_n &= \{i \in \mathbb{N} : q_i \in B(0,n)\}, \\ F_i^n(t) &= \sum_{\substack{j: j \neq i, \\ j \in I_n}} F(q_i^n(t) - q_j^n(t)) \end{aligned}$$

and

$$X^n(t) = \{q_i^n(t), v_i^n(t)\}_{i \in I_n}$$

is the time evolved finite configuration.

Even if in this paper we consider the more general case of long-range potentials, it is useful to underline the differences that occur considering short-range and long-range potentials (in both cases of a superstable, bounded type). For short-range potentials the following theorem holds.

Theorem 2.1. If $X \in \mathcal{X}_\xi$, there exists a unique flow $t \to X(t)$, with $X(t) = \{q_i(t), v_i(t)\}_{i \in \mathbb{N}} \in \mathcal{X}_{\frac{3}{2}\xi}$, satisfying:

$$\ddot{q}_i(t) = F_i(X(t)), \quad X(0) = X. \tag{2.11}$$

Moreover, $\forall t > 0$ and $\forall i \in \mathbb{N}$,

$$\lim_{n \to +\infty} q_i^n(t) = q_i(t), \qquad \lim_{n \to +\infty} v_i^n(t) = v_i(t). \tag{2.12}$$

Remark. In ref. 2 also initial data contained in $\mathcal{X}_\xi$ evolve in $\mathcal{X}_{\frac{3}{2}\xi}$ (in which ξ was taken equal to 1).

For long-range potentials the existence of the dynamics is defined starting from conditions for which ξ is not too large: $\xi < 4/9$ (this restriction for ξ will be clear in Section 5, where it will be used to make the iterative method work). In order to include states of physical interest we then take $\xi \in [1/3, 4/9)$. The theorem in this case is the following.

Theorem 2.2. If $X \in \mathcal{X}_\xi$, there exists a unique flow $t \to X(t)$, with $X(t) = \{q_i(t), v_i(t)\}_{i\in\mathbb{N}} \in \tilde{\mathcal{X}}_\xi$, satisfying:

$$\ddot{q}_i(t) = F_i(X(t)), \qquad X(0) = X, \tag{2.13}$$

where

$$\tilde{\mathcal{X}}_\xi = \mathcal{X}_{\frac{3}{2}\xi} \cap \bar{\mathcal{X}}_\xi, \tag{2.14}$$

and

$$\bar{\mathcal{X}}_\xi = \{q_i, v_i : \forall i \in \mathbb{N} \ \ |v_i| \leqslant C\psi_\xi^{3/2}(|q_i|)\}, \tag{2.15}$$

with $C > 0$.
Moreover, $\forall t > 0$ and $\forall i \in \mathbb{N}$,

$$\lim_{n\to+\infty} q_i^n(t) = q_i(t), \qquad \lim_{n\to+\infty} v_i^n(t) = v_i(t). \tag{2.16}$$

Theorems 2.1 and 2.2 are the main results and their proofs occupy the rest of the present paper. The proofs are based on several steps: we introduce a mollified version of the local energy and we study its evolution in time under the partial dynamics. The energy conservation allows to prove that the local energy grows in time at most as the cube of the maximal velocity of the particles. On the other hand a suitable time average allows to control the maximal velocity via the local energy in a good way. The result is achieved by letting $n \to \infty$. The philosophy of the proof is similar to that of ref. 2. Actually in that paper the authors use many times the positivity and the finite range of the interaction, while in the present paper the interaction can be negative and with long-range behavior. This fact requires a new mollifier and other cumbersome technical tools.

In the sequel we will need to split the potential into two terms: a short-range one, $\phi^{(1)}$, and a long-range one, $\phi^{(2)}$.

To do so, let us take, for $r > \max(r_0, \sqrt{3})$:

$$\begin{aligned} \phi(x) &= \phi^{(1)}(x) + \phi^{(2)}(x), \\ \phi^{(1)}(x) &= \phi(x)\, \chi(|x| \leqslant r), \\ \phi^{(2)}(x) &= \phi(x) - \phi^{(1)}(x), \\ |\phi^{(2)}(x)| &\leqslant \frac{G_1}{|x|^\gamma}. \end{aligned} \tag{2.17}$$

The following proposition holds.

Proposition 2.1. Let ϕ as in (2.17). Then $\exists \bar{r} > 0$ such that, $\forall r \geqslant \bar{r}$, $\phi^{(1)}$ is superstable.

Proof. From the superstability of ϕ we have

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \phi_{i,j} = \frac{1}{2}\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \left(\phi_{i,j}^{(1)} + \phi_{i,j}^{(2)}\right) \geqslant -B\,n + A \sum_{i\in\mathbb{Z}^3} n_{\Delta_i}^2 \\ &\quad\Rightarrow \frac{1}{2}\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \phi_{i,j}^{(1)} \geqslant -\frac{1}{2}\sum_{i,j}^{\star} \frac{G_1}{|q_i - q_j|^\gamma} - B\,n + A \sum_{i\in\mathbb{Z}^3} n_{\Delta_i}^2, \end{aligned} \tag{2.18}$$

where $\sum_{i,j}^{\star}$ is the sum restricted to particles at distance greater than r. Let us consider the first term on the right:

$$\begin{aligned} \sum_{i,j}^{\star} \frac{G_1}{|q_i - q_j|^\gamma} &= G_1 \sum_{k=1}^{\infty} \sum_{i\neq j} \chi(kr < |q_i - q_j| \leqslant (k+1)r) \frac{1}{|q_i - q_j|^\gamma} \\ &\leqslant G_1 \sum_{k=1}^{\infty} \frac{1}{(rk)^\gamma} \sum_{i\neq j} \chi(kr < |q_i - q_j| \leqslant (k+1)r) \\ &\leqslant \sum_{k=1}^{\infty} \sum_{\substack{l\in\mathbb{Z}^3,\\ m\in\mathbb{Z}^3}} \chi(kr - \sqrt{3} \leqslant |l - m| < (k+1)r + \sqrt{3}) \frac{G_1}{(kr)^\gamma} n_{\Delta_l} n_{\Delta_m} \\ &\leqslant \sum_{i\in\mathbb{Z}^3} n_{\Delta_i}^2 \sum_{k=1}^{\infty} \frac{G_1}{(rk)^\gamma} \\ &\qquad \times \mathbf{Card}\{\mathbb{Z}^3 \cap (B(0, (k+1)r + \sqrt{3}) \setminus B(0, kr - \sqrt{3}))\} \\ &\leqslant \frac{D_1}{r^{\gamma-3}} \sum_{k=1}^{\infty} \frac{1}{k^{\gamma-2}} \sum_{i\in\mathbb{Z}^3} n_{\Delta_i}^2 \leqslant \frac{D_2}{r^{\gamma-3}} \sum_{i\in\mathbb{Z}^3} n_{\Delta_i}^2, \end{aligned} \tag{2.19}$$

when $\gamma > 3$.

Inserting (2.19) in (2.18), we obtain:

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\phi_{i,j}^{(1)} \geqslant \left(-\frac{D_2}{2\,r^{\gamma-3}}+A\right)\sum_{i\in\mathbb{Z}^3} n_{\Delta_i}^2 - B\,n, \tag{2.20}$$

then for

$$r \geqslant \bar{r} = \max\left(\left(\frac{2D_2}{A}\right)^{1/(\gamma-3)}, r_0, \sqrt{3}\right) \tag{2.21}$$

we obtain the thesis:

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\phi_{i,j}^{(1)} \geqslant \frac{3}{4}A\sum_{i\in\mathbb{Z}^3} n_{\Delta_i}^2 - B\,n. \quad \blacksquare \tag{2.22}$$

For a configuration X with finite cardinality, let us define a mollified version of the energy (plus b times the number of particles, with $b > B$) for the particles contained into the ball $B(\mu, R)$, by means of a suitable weight-function:

$$W(X;\mu,R) = \sum_{i\in\mathbb{N}} f_i^{\mu,R}\Big(\frac{v_i^2}{2} + \frac{1}{2}\sum_{j:j\neq i}\phi_{i,j} + b\Big) \tag{2.23}$$

with a weight-function

$$f_i^{\mu,R} \equiv f(q_i-\mu, R) \equiv \int_{\mathbb{R}^3}\theta\left(\frac{|q_i-\mu-y|}{R}\right)\left(\frac{1}{1+\alpha|y|}\right)^{\lambda} dy, \tag{2.24}$$

where θ: $\mathbb{R}^+ \to (0,1]$, is continuously differentiable and it is such that

1. $\theta(x) = (1+\alpha x)^{-\lambda}$ for $x \geqslant 2$,

2. $\theta(x)$ is concave for $x \leqslant 2$,

3. $\theta(x) = \theta(2) - \frac{1}{2}\theta'(2)$, for $x \leqslant 1$.

Notice that

$$\theta(x) \leqslant (1+\alpha x)^{-\lambda} \tag{2.25}$$

and

$$|\theta^{'}(x)| \leqslant \lambda\,\alpha(1+\alpha x)^{-(\lambda+1)} \tag{2.26}$$

with $\lambda > 3$ and $\alpha \in (0, 1]$. In the sequel we shall assume $\lambda \in (4, \gamma - 3)$ and α small enough (for details see Appendix A).

Following,[1] let us show the main properties of the weight-function:

Proposition 2.2. There exist positive constants C_1, C_2, depending only on α and λ, such that, for any $R > 1$, the following properties hold

1. $f(x, R) \leqslant C_1(1+\alpha|x|/R)^{-\lambda}$,

2. $f(x, R) \geqslant C_2(1+\alpha\,|x|/R)^{-\lambda}$,

3. $f(x, R) \leqslant (1+\alpha|x-y|)^{\lambda} f(y, R)$.

Proof. 1. Let us prove the first property. Multiplying $f(x, R)$ by $(1+\alpha|x|/R)^{\lambda}$, using the triangular inequality we obtain:

$$\begin{aligned}\left(1+\alpha\frac{|x|}{R}\right)^{\lambda} f(x, R) &\leqslant \int_{\mathbb{R}^3} dy \left(\frac{R+\alpha|y|+\alpha|x-y|}{R+\alpha|x-y|}\right)^{\lambda} \frac{1}{(1+\alpha|y|)^{\lambda}} \\ &\leqslant 2^{\lambda}\int_{\mathbb{R}^3} dy\, \frac{(1+\alpha|y|)^{\lambda}+(R+\alpha|x-y|)^{\lambda}}{(R+\alpha|x-y|)^{\lambda}}\, \frac{1}{(1+\alpha|y|)^{\lambda}},\end{aligned}$$

considering that $\forall a, b \in \mathbb{R}^+$ it holds $(a+b)^{\lambda} \leqslant 2^{\lambda}\,(a^{\lambda}+b^{\lambda})$. Then

$$\begin{aligned}\left(1+\alpha\frac{|x|}{R}\right)^{\lambda} f(x, R) &\leqslant 2^{\lambda}\int_{\mathbb{R}^3} dy\, \frac{1}{(1+\alpha|y|)^{\lambda}} + 2^{\lambda}\int_{\mathbb{R}^3} dy\, \frac{1}{(R+\alpha|y-x|)^{\lambda}} \\ &\leqslant 2^{\lambda+1}\int_{\mathbb{R}^3} dy\, \frac{1}{(1+\alpha|y|)^{\lambda}} \leqslant C_1\end{aligned}$$

for $R > 1$.

2. Notice first that:

$$\theta\left(\frac{|x-y|}{R}\right) \geqslant \theta(2)\,\frac{1}{(1+\alpha|x-y|/R)^{\lambda}}\,, \tag{2.27}$$

being $\theta(2)=\min_{|x|\leqslant 2}\theta(x)$. So for the weight-function we have

$$f(x,R)\geqslant \theta(2)\int_{\mathbb{R}^3} dy\,\frac{1}{(1+\alpha|x-y|/R)^{\lambda}}\,\frac{1}{(1+\alpha|y|)^{\lambda}}\,.$$

Multiplying $f(x,R)$ by $(1+\alpha|x|/R)^{\lambda}$, we obtain:

$$\begin{aligned}\left(1+\alpha\frac{|x|}{R}\right)^{\lambda} f(x,R) &\geqslant \frac{\theta(2)}{2^{\lambda}}\int_{\mathbb{R}^3} dy\,\frac{1}{(1+\alpha|y|)^{\lambda}}\,\frac{(1+\alpha|x|/R)^{\lambda}}{(1+\alpha|y|/R)^{\lambda}+(1+\alpha|x|/R)^{\lambda}}\\ &\geqslant \frac{\theta(2)}{2^{\lambda}}\int_{\mathbb{R}^3} dy\,\frac{1}{(1+\alpha|y|)^{\lambda}}\,\frac{1}{1+\left(\frac{1+\alpha|y|/R}{1+\alpha|x|/R}\right)^{\lambda}}\\ &\geqslant \frac{\theta(2)}{2^{\lambda}}\int_{\mathbb{R}^3} dy\,\frac{1}{(1+\alpha|y|)^{\lambda}}\,\frac{1}{1+(1+\alpha|y|/R)^{\lambda}}\\ &\geqslant \frac{\theta(2)}{2^{\lambda}}\int_{\mathbb{R}^3} dy\,\frac{1}{(1+\alpha|y|)^{\lambda}}\,\frac{1}{1+(1+\alpha|y|)^{\lambda}}\geqslant C_2\end{aligned}$$

for $R>1$.

3. For the third relation let us write the function f in the following way, putting $x-y=z$:

$$f(x,R)=\int_{\mathbb{R}^3}\theta\left(\frac{|z|}{R}\right)\left(\frac{1}{1+\alpha|x-z|}\right)^{\lambda} dz.$$

Since

$$\frac{1}{1+\alpha|x-z|}\leqslant\frac{1+\alpha|x-y|}{1+\alpha|y-z|},$$

the thesis follows (last inequality becomes evident multiplying both sides by $(1+\alpha|x-z|)(1+\alpha|y-z|)$ and using the triangular inequality). ∎

The choice of such a weight-function will be evident later, in the proof of Lemma 3.1. This function, unlike the mollifier function used in ref. 2, allows also to give some superstability estimates for the energy of a bounded region of the space, essential in the proof of Lemma 3.2.

Notice that, if the interaction has finite range, we could use an explicit weight-function, i.e. $f(x)=1/\cosh(x)$. In general an exponential decay for the weight-function is too fast for taking into account potentials with a power-law decay.

We give now a short explanation for the technical assumption on the power-law decay ($\gamma>7$) of the interaction. The weight-function must decay slower than the interaction ($\gamma>3+\lambda$) to handle the border terms of the mollified energy (see (A.12)); moreover the weight-function must decay fast enough ($\lambda>4$, see (C.9)) to obtain the boundedness of the mollified density energy $W_\xi(X)$ defined in (3.4).

3. PROPERTIES OF THE MOLLIFIED ENERGY

We present here a lemma, whose proof is shown in Appendix A, that gives a superstability property of the mollified energy.

Lemma 3.1. There exist $C_3>0$ and $\bar{\alpha}\in(0,1)$, not depending on R, such that $\forall\alpha\in(0,\bar{\alpha})$:

$$W(X;\mu,R)\geqslant C_3\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\,n^2_{\Delta_k}. \tag{3.1}$$

We actually prove a stronger condition:

$$\begin{aligned} W(X;\mu,R) &\geqslant \sum_{i\in\mathbb{N}} f_i^{\mu,R}\Big(\frac{1}{2}\sum_{j:j\neq i}\phi_{i,j}+b\Big)\\ &\geqslant C_3\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\,n^2_{\Delta_k}\geqslant 0, \end{aligned} \tag{3.2}$$

which implies that the interaction energy is non-negative. In the sequel the parameter α appearing in Lemma 3.1 will be considered fixed.

From Lemma 3.1 we can derive the following corollaries:

Corollary 3.1. There exist $C_3, C_4>0$, not depending on R, such that:

$$\begin{aligned} C_3\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\,n^2_{\Delta_k}\leqslant W(X;\mu,R)\leqslant C_4 &\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\,n^2_{\Delta_k}\\ &+\sum_{i\in\mathbb{N}} f_i^{\mu,R}\,\frac{v_i^2}{2}. \end{aligned} \tag{3.3}$$

The first inequality reproduces Lemma 3.1, while the proof of the second inequality will be given in Appendix B.

The function W is a technical tool. The following Corollary (whose proof is given in Appendix C) shows the relation with the initial data. Defining

$$W_\xi(X) \equiv \sup_\mu \sup_{R>\psi_\xi(|\mu|)} \frac{W(X;\mu,R)}{R^3}, \tag{3.4}$$

then it holds:

Corollary 3.2. $\exists\, C_5, C_6 > 0$, not depending on R, such that:

$$C_5\, Q_\xi(X) \leqslant W_\xi(X) \leqslant C_6\, Q_\xi(X). \tag{3.5}$$

We can give now an estimate for the mollified energy, useful for the proof of the existence of the dynamics.

Lemma 3.2. For $X \in \mathcal{X}_\xi$, there exists a positive constant C_7 such that

$$\sup_\mu W(X^n(t);\mu, R(n,t)) \leqslant C_7 R^3(n,t)\,, \tag{3.6}$$

where

$$R(n,t) = \varphi(n) + \int_0^t ds\, V^n(s) \tag{3.7}$$

with

$$\varphi(n) = \psi_\xi^{3/2}(n)$$

and

$$V^n(s) = \max_{i\in I_n} \left\{ \sup_{0\leqslant\tau\leqslant s} |v_i^n(\tau)| \right\}.$$

Proof. For $0 \leqslant s \leqslant t \leqslant T$ let us define

$$R(n,t,s) = R(n,t) + \int_s^t V^n(\tau)\, d\tau\,. \tag{3.8}$$

Notice that

$$\dot{R}(n,t,s) \equiv \frac{\partial R}{\partial s}(n,t,s) = -V^n(s) \leqslant 0,$$

moreover

$$R(n,t,t)=R(n,t), \quad R(n,t,0)<2R(n,t).$$

Let us derive with respect to s the quantity:

$$\begin{aligned} W(X^n(s);\mu,R(n,t,s)) &= \sum_{i\in\mathbb{N}} f_i^{\mu,R(n,t,s)} \left(\frac{v_i^2}{2} + \frac{1}{2} \sum_{\substack{j:j\neq i,\\ j\in\mathbb{N}}} \phi_{i,j} + b \right) \\ &= \sum_{i\in\mathbb{N}} f_i^{\mu,R(n,t,s)}\, w_i \end{aligned} \tag{3.9}$$

with

$$w_i \equiv \frac{v_i^2}{2} + \frac{1}{2} \sum_{j:j\neq i} \phi_{i,j} + b. \tag{3.10}$$

We have:

$$\frac{\partial W}{\partial s} = \dot{W}_1 + \dot{W}_2, \tag{3.11}$$

where

$$\begin{aligned} \dot{W}_1 &\equiv \sum_i w_i \int_{\mathbb{R}^3} dy\, \theta' \left(\frac{|q_i-\mu-y|}{R} \right) \\ &\quad \times \frac{1}{(1+\alpha|y|)^{\lambda}} \left(\frac{\mathrm{Vers}(q_i-y-\mu)\cdot v_i}{R(n,t,s)} - \frac{\dot{R}(n,t,s)}{R^2(n,t,s)} |q_i-y-\mu| \right), \\ \dot{W}_2 &\equiv \sum_{i\neq j} f_i^{\mu,R(n,t,s)} \left(v_i \cdot F_{i,j} - \frac{1}{2} F_{i,j}\cdot (v_i - v_j) \right). \end{aligned} \tag{3.12}$$

We have denoted by $\mathrm{Vers}(x)$ the versor of the vector $x\in\mathbb{R}^3$. Let us consider now the first term $\dot{W}_1$. Thanks to (2.26) and to the definition of V^n, we have:

$$\begin{aligned} |\dot{W}_1| \leqslant \lambda \left| \frac{\dot{R}}{R} \right| \alpha \sum_i |w_i| \int_{\mathbb{R}^3} dy\, \frac{1}{(1+\alpha|q_i-\mu-y|/R)^{\lambda+1}} \\ \times \frac{1}{(1+\alpha|y|)^{\lambda}} \left(1 + \frac{|q_i-y-\mu|}{R(n,t,s)} \right) \end{aligned}$$

$$\leqslant D_3 \left|\frac{\dot R}{R}\right| \sum_i |w_i| \int_{\mathbb{R}^3} dy \, \frac{1}{(1+\alpha|q_i-\mu-y|/R)^\lambda} \, \frac{1}{(1+\alpha|y|)^\lambda}$$

$$\leqslant D_4 \left|\frac{\dot R}{R}\right| \sum_i f_i^{\mu,R(n,t,s)} |w_i|, \tag{3.13}$$

where in the last inequality we have applied (2.27). From the positivity of the mollified energy and from estimates analogous to those used to obtain (B.1) we have:

$$|\dot W_1| \leqslant D_5 \left|\frac{\dot R}{R}\right| \left(W(X;\mu,R) + \sum_{i\in\mathbb{Z}^3} n^2_{\Delta_i} f(|i-\mu|,R) \right) \tag{3.14}$$

and from Lemma 3.1 we obtain:

$$|\dot W_1| \leqslant D_6 \left|\frac{\dot R}{R}\right| W(x;\mu,R). \tag{3.15}$$

For the second term $\dot W_2$ we are going to give also an estimate of the form:

$$\dot W_2 \leqslant D_{11} \left|\frac{\dot R}{R}\right| W(X;\mu,R). \tag{3.16}$$

Let us evaluate

$$\begin{aligned}\dot W_2 &= \sum_{i\neq j} f_i^{\mu,R(n,t,s)} \left(v_i \cdot F_{i,j} - \frac{1}{2} F_{i,j} \cdot (v_i - v_j) \right) \\ &= \frac{1}{2} \sum_{i\neq j} f_i^{\mu,R(n,t,s)} F_{i,j} (v_i + v_j). \end{aligned} \tag{3.17}$$

Since $F_{i,j} = -F_{j,i}$, it results:

$$\begin{aligned}\dot W_2 &= \frac{1}{2} \sum_{i\neq j} f_i^{\mu,R(n,t,s)} F_{i,j} \cdot (v_i + v_j) \\ &= \frac{1}{2} \sum_{i\neq j} f_i^{\mu,R(n,t,s)} F_{i,j} \cdot v_i - \frac{1}{2} \sum_{i\neq j} f_j^{\mu,R(n,t,s)} F_{i,j} \cdot v_i \\ &= -\frac{1}{2} \sum_{i\neq j} (f_i^{\mu,R(n,t,s)} - f_j^{\mu,R(n,t,s)}) \nabla\phi_{i,j} \cdot v_i(s). \end{aligned} \tag{3.18}$$

Let us estimate now the addends of the sum one by one. From the properties of $\theta(x)$ (2.26), (2.27) and of the potential we have:

$$|f_i^{\mu,R(n,t,s)} - f_j^{\mu,R(n,t,s)}| \leqslant D_7 \frac{|q_i - q_j|}{R(n,t,s)} (f_i^{\mu,R(n,t,s)} + f_j^{\mu,R(n,t,s)}). \quad (3.19)$$

Being

$$|\nabla\phi(|q_i - q_j|)| \leqslant D_8\big(1 + |q_i - q_j|\big)^{-\gamma-1}, \quad (3.20)$$

then, using an estimate analogous to (B.1):

$$\begin{aligned} |\dot{W}_2| &\leqslant D_9 \left|\frac{\dot{R}}{R}\right| \sum_{i\in\mathbb{N}} \sum_{\substack{j\in\mathbb{N}:\\ i\neq j}} (f_i^{\mu,R(n,t,s)} + f_j^{\mu,R(n,t,s)}) \frac{1}{\big(1 + |q_i - q_j|\big)^{\gamma}} \\ &\leqslant D_{10} \left|\frac{\dot{R}}{R}\right| \sum_{i\in\mathbb{Z}^3} f(|i-\mu|, R)\, n_{\Delta_i}^2. \end{aligned} \quad (3.21)$$

Using Lemma 3.1 we close the estimate with the function W:

$$|\dot{W}_2| \leqslant D_{11} \left|\frac{\dot{R}(n,t,s)}{R(n,t,s)}\right| W(X^n(s); \mu, R(n,t,s)). \quad (3.22)$$

We have so proved that

$$\left|\frac{\partial W(X^n(s); \mu, R(n,t,s))}{\partial s}\right| \leqslant D_{12} \left|\frac{\dot{R}(n,t,s)}{R(n,t,s)}\right| W(X^n(s); \mu, R(n,t,s)). \quad (3.23)$$

Integrating we have

$$\begin{aligned} W(X^n(s); \mu, R(n,t,s)) \leqslant W\ &(X^n(0); \mu, R(n,t,0)) \\ &+ D_{12} \int_0^s d\tau \left|\frac{\dot{R}(n,t,\tau)}{R(n,t,\tau)}\right| W(X^n(\tau); \mu, R(n,t,\tau)). \end{aligned}$$

Let us use now the Gronwall's lemma to handle the previous inequality

$$W(X^n(s); \mu, R(n,t,s)) \leqslant W(X^n(0); \mu, R(n,t,0)) \left(\frac{R(n,t,0)}{R(n,t,s)}\right)^{D_{12}}, \quad (3.24)$$

from which, being $\frac{R(n,t,0)}{R(n,t,s)} \leqslant 2$, we obtain

$$W(X^n(s); \mu, R(n,t,s)) \leqslant 2^{D_{12}} W(X^n(0); \mu, R(n,t,0)), \tag{3.25}$$

and since $R(n,t,t) = R(n,t)$, taking the supremum over μ, we have

$$\sup_\mu W(X^n(t); \mu, R(n,t)) \leqslant D_{13} \sup_\mu W(X^n(0); \mu, R(n,t,0)). \tag{3.26}$$

From Corollary 3.2 and by the hypothesis on the initial data, being $R(n,t,0) > \psi_\xi(n)$, we get

$$\sup_\mu W(X^n(0); \mu, R(n,t,0)) \leqslant C_6 Q_\xi(X) R^3(n,t,0), \tag{3.27}$$

thus

$$\sup_\mu W(X^n(t); \mu, R(n,t)) \leqslant D_{14} R^3(n,t). \quad \blacksquare$$

In Lemma 3.3 we present some relations that will be used in the sequel. The proof is in Appendix D.

Lemma 3.3. Let X be a configuration with finite cardinality. Then, for any $R > 1$ there exist positive constants C_8, C_9, C_{10}, C_{11} such that

(i) if $n \in \mathbb{N}$, $n > 1$

$$W(X; \mu, nR) \leqslant C_8 n^\lambda W(X; \mu, R); \tag{3.28}$$

(ii) if $n \in \mathbb{N}$, $n > 1$

$$W(X; \mu, R) \leqslant C_9 W(X; \mu, nR); \tag{3.29}$$

(iii) $$N(X, \mu, R) \equiv \sum_i \chi(|q_i - \mu| < R) \leqslant C_{10} R^{3/2} W(X; \mu, R)^{1/2}; \tag{3.30}$$

(iv) for $0 < \rho < R$

$$\sum_{i \neq j} \chi(|q_i - q_j| < \rho) \chi(|q_i - \mu| < R) \chi(|q_j - \mu| < R) \leqslant C_{11} \rho^3 W(X; \mu, R). \tag{3.31}$$

We will need an estimate for the force F_i that at time t acts on the particle i:

$$F_i(X^n(t)) \equiv -\sum_{j\in I_n} \nabla\phi(|q_i(t)-q_j(t)|). \tag{3.32}$$

We can make the following decomposition:

$$|F_i(X^n(t))| \leqslant F_i^{(1)} + F_i^{(2)},$$

where $F_i^{(1)}$ represents a bound for the absolute value of the force acting on the particle i, due to the particles j contained in $B(q_i(t), r)$, with r not less than $\bar{r}$, defined in Proposition 2.1, and $F_i^{(2)}$ is a bound for the absolute value of the force acting on the particle i, due to the particles j contained in $B^c(q_i(t), r)$.

Using the third property of Lemma 3.3, the first term is bounded by:

$$\begin{aligned} F_i^{(1)} &\leqslant \|F\|_\infty N(X^n(t), q_i(t), r) \leqslant \|F\|_\infty C_{10}\, r^{3/2} W(X^n(t); q_i(t), r)^{1/2} \\ &\leqslant \|F\|_\infty D_{15}\, r^{3/2} \sup_\mu W(X^n(t); \mu, R(n,t))^{1/2} \leqslant D_{16}\, R^{3/2}(n,t), \end{aligned}$$

where, for sufficiently large n, we have used Lemma 3.2.

Let us give now a bound for the second term; for $R = R(n,t) \gg r$ we have

$$\begin{aligned} F_i^{(2)} \leqslant\ & G_2 \sum_{j:|q_i-q_j|>r} \frac{1}{|q_i-q_j|^{\gamma+1}} \\ \leqslant\ & G_2 \sum_{k=1}^{[R/r]+1} \sum_j \chi(kr < |q_i-q_j| \leqslant (k+1)r) \frac{1}{(kr)^{\gamma+1}} \\ &+ G_2 \sum_{k=1}^{+\infty} \sum_j \chi(kR < |q_i-q_j| \leqslant (k+1)R) \frac{1}{(kR)^{\gamma+1}} \\ \leqslant\ & G_2 \sum_{k=1}^{[R/r]+1} N(X, q_i, (k+1)r)) \frac{1}{(rk)^{\gamma+1}} \\ &+ G_2 \sum_{k=1}^{\infty} N(X, q_i, (k+1)R)) \frac{1}{(kR)^{\gamma+1}} \end{aligned}$$

$$
\begin{aligned}
&\leqslant D_{17} \sum_{k=1}^{[R/r]+1} ((k+1)r)^{3/2}\, W^{1/2}(X, q_i, (k+1)r)\frac{1}{(rk)^{\gamma+1}}\\
&\quad + \; D_{17} \sum_{k=1}^{\infty}((k+1)R)^{3/2}\, W^{1/2}(X, q_i, (k+1)R)\frac{1}{(Rk)^{\gamma+1}}\\
&\leqslant D_{18} \left(R^{3/2} + R^{3/2-\gamma-1} \sup_{\mu} W^{1/2}(X, \mu, R) \sum_{k=1}^{\infty} k^{\lambda/2}\frac{1}{k^{\gamma+1-3/2}}\right)\\
&\leqslant D_{19} \left(R^{3/2} + R^{3-\gamma-1} \sum_{k=1}^{+\infty}\frac{1}{k^{\gamma-\lambda/2-1/2}}\right),
\end{aligned}
\tag{3.33}
$$

where in the penultimate line we have used the first property of Lemma 3.3. Since $\gamma > 3+\lambda$ we obtain:

$$F_i^{(2)} \leqslant D_{20} R^{3/2}(n,t). \tag{3.34}$$

Then

$$|F_i(X^n(t))| \leqslant D_{21}\, R^{3/2}(n,t). \tag{3.35}$$

In the proof of Proposition 4.1 we will need an estimate for the force, $|\bar{F}_i|$, due to the particles j at distance larger than $R(n,t)^{1/4}$ from the particle i:

$$
\begin{aligned}
|\bar{F}_i| &\leqslant G_2 \sum_{j:|q_i-q_j|>R^{1/4}} \frac{1}{|q_i-q_j|^{\gamma+1}}\\
&\leqslant G_2 \sum_{k=1}^{+\infty}\sum_{j} \chi(kR^{1/4} \leqslant |q_i-q_j| < (k+1)R^{1/4})\frac{1}{(kR^{1/4})^{\gamma+1}}\\
&\leqslant G_2 \sum_{k=1}^{\infty} N(X, q_i, (k+1)R^{1/4}))\frac{1}{(kR^{1/4})^{\gamma+1}}\\
&\leqslant D_{22} \sum_{k=1}^{\infty}((k+1)R^{1/4})^{3/2}\, W^{1/2}(X, q_i, (k+1)R)\frac{1}{(kR^{1/4})^{\gamma+1}}\\
&\leqslant D_{23}\, R^{3/8-(\gamma+1)/4} \sup_{\mu} W^{1/2}(X, \mu, R) \sum_{k=1}^{\infty} k^{\lambda/2}\frac{1}{k^{\gamma+1-3/2}}\\
&\leqslant D_{24} R^{3/8+3/2-(\gamma+1)/4} \sum_{k=1}^{+\infty}\frac{1}{k^{\gamma-\lambda/2-1/2}}.
\end{aligned}
\tag{3.36}
$$

4. DYNAMICAL ESTIMATES

The following two propositions give bounds on the maximal velocity of a particle and on the work done by the system over a single particle.

In this section, we shall omit any explicit notational dependence on n for $R(n,t)$ and $\{q_i^n(t), v_i^n(t)\}$ for simplicity, since, from now on, n will be fixed.

Proposition 4.1. For any positive $T < +\infty$, there exists a positive constant C_{12} such that, for $t \leqslant T$,

$$V^n(t) \leqslant C_{12}\, R(t), \tag{4.1}$$

where

$$R(t) = \varphi(n) + \int_0^t V^n(s)\, ds, \tag{4.2}$$

and

$$\varphi(n) = \psi_\xi^{3/2}(n). \tag{4.3}$$

Proposition 4.2. For $0 \leqslant s \leqslant t \leqslant T$ and any $\zeta \in [1/2, 1]$, we set:

$$\Delta = \zeta\, R(t)^{-4/6}. \tag{4.4}$$

Suppose that, for some $i \in I_n$ and some suitable constant $\bar{A} > 1$:

$$\inf_{\tau \in [s-\Delta, s]} |\, v_i(\tau)\,| = \bar{A}\, R(t). \tag{4.5}$$

Then there exists a constant C_{13} independent of $\bar{A}$ such that:

$$\left| \int_{s-\Delta}^{s} d\tau \sum_j v_j \cdot F_{i,j} \right| \leqslant C_{13} \Delta R(t)^2 \tag{4.6}$$

Using Proposition 4.2 we are able to prove Proposition 4.1.

Proof of Proposition 4.1. The proof will be achieved by contradiction. We first notice that, by the initial conditions, $V^n(0) \leqslant Q_\xi(X)^{1/2} \varphi(n) = Q_\xi(X)^{1/2} R(0)$ and then (4.1) is verified for $t = 0$.

Suppose that, for some $t^\star \in [0, t]$ and $i \in I_n$ we have:

$$V^n(t^\star) = |v_i(t^\star)| = \tilde{A}\, R(t) \tag{4.7}$$

for a suitable constant $\tilde{A}$ to be fixed later and satisfying $\tilde{A} > 2(Q_\xi(X)^{1/2} + 1)$. We also fix $t_1 \in [0, t^\star)$, such that

$$|v_i(t_1)| = (Q_\xi(X)^{1/2} + 1)R(t), \tag{4.8}$$

$$\inf_{\tau \in (t_1, t^\star)} |v_i(\tau)| \geqslant (Q_\xi(X)^{1/2} + 1)R(t) \tag{4.9}$$

and $|t^\star - t_1| = H\Delta$ for some integer $H \geqslant 1$ and a suitable choice of ζ. This can be done because by

$$v_1(t^\star) = v_i(t_1) + \int_{t_1}^{t^\star} F_i(X^n(\tau))\, d\tau \tag{4.10}$$

and by (3.35), we find

$$\tilde{A} R(t) \leqslant (Q_\xi(X)^{1/2} + 1)R(t) + D_{21}(t^\star - t_1)R(t)^{3/2} \tag{4.11}$$

and hence

$$(t^\star - t_1) \geqslant E_1\, R(t)^{-1/2} \gg R(t)^{-4/6}, \tag{4.12}$$

therefore, for a suitable choice of $\zeta \in [1/2, 1]$, $\frac{R(t)^{4/6}|t^\star - t_1|}{\zeta}$ is integer.

Furthermore, defining the set

$$\bar{Y}_n = \{j \in I_n : |q_i(\tau) - q_j(\tau)| \leqslant R(t)^{1/4} \text{ for some } \tau \in [t_1, t^*]\}, \tag{4.13}$$

we have

$$\begin{aligned} \frac{1}{2}v_i^2(t^\star) - \frac{1}{2}v_i^2(t_1) &= \int_{t_1}^{t^\star} ds \sum_j v_i \cdot F_{i,j} \\ &= \mathcal{L}_1 + \mathcal{L}_2, \end{aligned} \tag{4.14}$$

where

$$\mathcal{L}_1 \equiv \int_{t_1}^{t^\star} ds \sum_{j \in \bar{Y}_n^c} v_i \cdot F_{i,j} \quad \text{and} \quad \mathcal{L}_2 \equiv \int_{t_1}^{t^\star} ds \sum_{j \in \bar{Y}_n} v_i \cdot F_{i,j}. \tag{4.15}$$

For $\mathcal{L}_1$ we have:

$$|\mathcal{L}_1| \leqslant \max_{s\in[t_1,t^*]}\left(\sum_{j\in\bar{Y}_n^c}|F_{i,j}(s)|\right)\int_{t_1}^{t^\star} ds\,|v_i(s)| \leqslant E_2 R(t)^{\frac{23}{8}-\frac{\gamma+1}{4}} \tag{4.16}$$

where the time integral is bounded by $R(t)$ (see (3.7)), and for the sum of the force we have used (3.36). Equation (4.16) clearly gives $|\mathcal{L}_1| \leqslant E_3\, R(t)^2$. Let us consider the second term $\mathcal{L}_2$:

$$\begin{aligned}\mathcal{L}_2 &= \int_{t_1}^{t^\star} ds \sum_{j\in\bar{Y}_n}(v_i - v_j)\cdot F_{i,j} + \sum_{h=1}^{H}\int_{t_1+(h-1)\Delta}^{t_1+h\Delta} ds \sum_{j\in\bar{Y}_n} v_j\cdot F_{i,j}\\ &= -\sum_{j\in\ \bar{Y}_n}\phi(q_i(t^\star)-q_j(t^\star)) + \sum_{j\in\bar{Y}_n}\phi(q_i(t_1)-q_j(t_1))\\ &\quad + \sum_{h=1}^{H}\int_{t_1+(h-1)\Delta}^{t_1+h\Delta} ds \sum_{j\in\bar{Y}_n} v_j\cdot F_{i,j}\end{aligned} \tag{4.17}$$

and, following a similar method to that used to obtain (3.35), we get

$$\left|\sum_{j\in\bar{Y}_n}\phi(q_i(t^\star)-q_j(t^\star))\right| \leqslant E_4 R(t)^{3/2}. \tag{4.18}$$

The same bound holds for $\sum_j \phi((q_i(t_1)-q_j(t_1))$. Thus, using Proposition 4.2 to control the last term of (4.17), we have:

$$\frac{1}{2}v_i^2(t^\star) \leqslant (Q_\xi(X)+1+E_3)R(t)^2 + 2E_4 R(t)^{3/2} + C_{13}R(t)^2\,|t^\star - t_1|, \tag{4.19}$$

hence

$$\tilde{A}^2\, R(t)^2 \leqslant 2\left(Q_\xi(X)+1+E_3+2E_4+C_{13}T\right)R(t)^2. \tag{4.20}$$

The above inequality can't be satisfied for any $\tilde{A}^2$ larger than $2(Q_\xi(X)+1+E_3+2E_4+C_{13}T)$. This clearly contradicts (4.7) (for this choice of $\tilde{A}$), therefore the proposition is proved. ▮

Proof of Proposition 4.2. Let us set

$$J = [s-\Delta, s], \tag{4.21}$$

$$Y_n = \{j\in I_n : |q_i(\tau)-q_j(\tau)| \leqslant R(t)^{1/4} \text{ for some } \tau\in J\}. \tag{4.22}$$

The particles belonging to Y_n^c can be easily handled: as shown in (4.16) we have

$$\left|\int_{s-\Delta}^{s} d\tau \sum_{j\in Y_n^c} v_j\cdot F_{i,j}\right| \leqslant E_2 R(t)^{\frac{23}{8}-\frac{\gamma+1}{4}} \leqslant E_5 R(t)^2\Delta, \tag{4.23}$$

being $\gamma>7$. Hence from now on we consider only the particles $j\in Y_n$. Let us split the set Y_n according to the following partition:

$$a_k=\{j\in Y_n : 2^{k-1}\,R(t)^{4/6}\leqslant \sup_{\tau\in J}|v_j(\tau)|<2^k\,R(t)^{4/6},\ k=1,\ldots,k_{\max}\}, \tag{4.24}$$

where $k_{\max}$ is the maximum integer for which

$$2^{k_{\max}}\leqslant \tfrac{1}{2}R(t)^{2/6}, \tag{4.25}$$

$$a_0=\{j\in Y_n : \sup\nolimits_{\tau\in J}|v_j(\tau)|<R(t)^{4/6}\}, \tag{4.26}$$

$$\tilde a=\bigcup_{k=1}^{k_{\max}} a_k, \tag{4.27}$$

$$\bar a = Y_n\setminus(a_0\cup\tilde a). \tag{4.28}$$

Therefore

$$\left|\int_{s-\Delta}^{s} d\tau \sum_{j\in Y_n} v_j\cdot F_{i,j}\right| = \left|\int_{s-\Delta}^{s} d\tau \left\{\sum_{j\in\bar a}+\sum_{j\in\tilde a}+\sum_{j\in a_0}\right\} v_j\cdot F_{i,j}\right| \tag{4.29}$$

and we give below a bound for each term of the previous equality.

First of all we give an upper bound for the cardinality of $\bar a$. If $j\in\bar a$

$$|v_j(t^*)|=\max_{\tau\in J}|v_j(\tau)|\geqslant\frac{1}{4}R(t), \tag{4.30}$$

then by (3.35),

$$|v_j(\tau)|\geqslant\frac{1}{4}R(t)-D_{21}\Delta\,R(t)^{3/2}\geqslant\frac{1}{4}R(t)-D_{21}\,R(t)^{5/6}\geqslant\frac{1}{8}R(t), \tag{4.31}$$

for n (and so for $R(t)$) large enough.

By definition $R(t)$ is larger than the maximal displacement that a particle can undergo during the time interval $[0,t]$, then all the particles with indices in Y_n must be contained into the ball $B(q_i(0), 3R(t))$. Thus it follows from (C.1), (3.28) and (3.6) that

$$\sum_{j\in\bar{a}} v_j^2(\tau) \leqslant 2Q(X^n(\tau); q_i(0), 3R(t)) \leqslant 2\widetilde{L}\, W(X^n(\tau); q_i(0), 3R(t))$$
$$\leqslant 2C_8 3^{\lambda}\, \widetilde{L} W(X^n(\tau); q_i(0), R(t)) \leqslant 2C_8\, 3^{\lambda} \widetilde{L}\, C_7 R(t)^3, \quad (4.32)$$

then, by (4.31):

$$\frac{1}{64}|\bar{a}|R(t)^2 \leqslant E_6 R(t)^3, \quad (4.33)$$

which implies

$$|\bar{a}| \leqslant 64\, E_6\, R(t). \quad (4.34)$$

As a consequence, we have

$$\left| \int_{s-\Delta}^{s} d\tau \sum_{j\in\bar{a}} v_j \cdot F_{i,j} \right| \leqslant \|F\|_\infty \int_{s-\Delta}^{s} d\tau \left(\sum_{j\in\bar{a}} |v_j|^2 \right)^{1/2} |\bar{a}|^{1/2}$$
$$\leqslant E_7 R(t)^{3/2}\, R(t)^{1/2} \Delta = E_7 R(t)^2\, \Delta. \quad (4.35)$$

Let us consider now the contribution of the set $\tilde{a}$. Let $l \in \mathbb{N}$ with $1 \leqslant l \leqslant l_{\max}$ and $l_{\max} = \left[R(t)^{1/4}\right]$. In this way, using the decreasing property (2.4), we get:

$$\left| \int_{s-\Delta}^{s} d\tau \sum_{j\in a_k} v_j \cdot F_{i,j} \right|$$
$$\leqslant E_8\, R(t)^{4/6}\, 2^k \sum_{j\in a_k} \left\{ \sum_{l=1}^{l_{\max}} \frac{1}{l^{\gamma+1}} \int_{s-\Delta}^{s} d\tau\, \chi_{i,j}^{(l)}(\tau) \right.$$
$$\left. + \frac{1}{\left[R(t)^{1/4}\right]^{(\gamma+1)}} \int_{s-\Delta}^{s} d\tau\, \chi\left(|q_i(\tau) - q_j(\tau)| > \left[R(t)^{1/4}\right] \right) \right\}$$

$$\leqslant E_8 R(t)^{4/6} 2^k \sum_{j\in a_k} \left\{ \sum_{l=1}^{l_{\max}} \frac{1}{l^\gamma} \int_{s-\Delta}^{s} d\tau\, \chi_{i,j}^{(l)}(\tau) + \frac{1}{\left[R(t)^{1/4}\right]^\gamma} \int_{s-\Delta}^{s} d\tau\, \chi\left(|q_i(\tau)-q_j(\tau)| > \left[R(t)^{1/4}\right]\right) \right\}, \quad (4.36)$$

where

$$\chi_{i,j}^{(l)}(\tau) = \chi(|q_i(\tau)-q_j(\tau)| \leqslant l).$$

Now we want to study the time integral $\int_{s-\Delta}^{s} d\tau\, \chi_{i,j}^{(l)}(\tau)$, with $1 \leqslant l \leqslant l_{\max}$. In order to estimate this integral, we notice that for n sufficiently large:

$$\begin{aligned} |v_i(\tau)-v_j(\tau)| &\geqslant \inf_{\tau\in J} |v_i(\tau)| - \sup_{\tau\in J} |v_j(\tau)| \\ &\geqslant R(t) - 2^{k_{\max}} R(t)^{4/6} \geqslant \frac{1}{2} R(t). \end{aligned} \quad (4.37)$$

Suppose that $|q_i(t_0)-q_j(t_0)| = l$ at time $t_0 \in [s-\Delta, s]$, with outgoing velocities (i.e. $(v_i(t_0)-v_j(t_0))\cdot(q_i(t_0)-q_j(t_0)) \geqslant 0$). Then we are going to prove that the pair (i, j), once reached a relative distance larger than l, it will never reach a distance smaller than l. Let $t_1 \in (s-\Delta, s)$ denote the time in which $(q_i(\tau)-q_j(\tau))^2$ reaches its maximum value, say r_1^2 (for this reason $(v_i(t_1)-v_j(t_1))\cdot(q_i(t_1)-q_j(t_1)) = 0$).

By the identity

$$\begin{aligned} \frac{1}{2}\frac{d^2}{d\tau^2}(q_i(\tau)-q_j(\tau))^2 &= (v_i(\tau)-v_j(\tau))^2 \\ &\quad + (q_i(\tau)-q_j(\tau))\cdot(F_i(\tau)-F_j(\tau)), \end{aligned}$$

and using (4.37), (3.35) we get:

$$(q_i(\tau)-q_j(\tau))^2 \geqslant r_1^2 + \frac{(\tau-t_1)^2}{2}\left(\frac{R(t)^2}{4} - D_{21}\, r_1 R(t)^{3/2}\right), \quad (4.38)$$

for $\tau > t_1$. By the definition of r_1 it follows that $r_1 \geqslant R(t)^{1/2}/(4\,D_{21})$, otherwise $(q_i(\tau)-q_j(\tau))^2 > r_1^2$. In this case

$$\frac{(\tau-t_1)^2}{2}\, r_1 R(t)^{3/2} \leqslant \frac{\Delta^2}{2}\, r_1 R(t)^{3/2} \leqslant \frac{\zeta^2}{2}\, r_1 R(t)^{1/6} \leqslant E_9 {r_1}^{4/3}. \quad (4.39)$$

Therefore

$$(q_i(\tau) - q_j(\tau))^2 \geqslant r_1^2 - E_9 {r_1}^{4/3} \gg l^2, \tag{4.40}$$

then the pair (i, j) will keep a relative distance larger than l in the time interval (t_0, s) (note that the last inequality clearly holds because $l_{max} = \left[R(t)^{1/4}\right] \ll R(t)^{1/2}$).

Now we repeat this argument when r_1 is the minimum distance between particles i and j; we again denote by t_1 the time in which this distance is reached. Supposing $r_1 < l$, we want to establish the exit time of the particle j from the ball $B(q_i(\tau), l)$; this time can be derived from the equation $(q_i(\tau) - q_j(\tau))^2 = l^2$, hence (4.38) implies

$$l^2 \geqslant r_1^2 + E_{10} \frac{(\tau - t_1)^2}{2} \frac{R(t)^2}{4},$$

$$(\tau - t_1)^2 \leqslant \frac{8(l^2 - r_1^2)}{E_{10} R(t)^2} \leqslant \frac{8l^2}{E_{10} R(t)^2} \Rightarrow |\tau - t_1| \leqslant \frac{E_{11}\, l}{R(t)}.$$

Thus

$$\int_{s-\Delta}^{s} \chi_{i,j}^{(l)}(\tau) \leqslant \frac{E_{12}\, l}{R(t)}. \tag{4.41}$$

In order to estimate the cardinality of a_k, we use again an upper bound of the energy as we have done for the set $\bar{a}$. Let be $\tau_j \in J$ such that $|v_j(\tau_j)| = \max_{\tau \in J} |v_j(\tau)|$. Thus

$$|a_k|\, 2^{2(k-1)}\, R^{8/6} \leqslant \sum_{j \in a_k} |v_j(\tau_j)|^2 \leqslant \sum_{j \in a_k} |v_j(s - \Delta)|^2 + \int_{s-\Delta}^{s} d\tau \sum_{j \in a_k} |v_j(\tau)| \sum_{p} |F_{p,j}(\tau)|. \tag{4.42}$$

Multiplying (4.42) by 2^{-k} and summing over k, we have

$$\sum_k \frac{1}{2} |a_k|\, 2^{(k-1)}\, R^{8/6} \leqslant \sum_k 2^{-k} \sum_{j \in a_k} |v_j(s - \Delta)|^2 + E_{13}\, R^{4/6} \int_{s-\Delta}^{s} d\tau \sum_{j \in \tilde{a}} \sum_{p} |F_{p,j}(\tau)|. \tag{4.43}$$

The latter term can be bounded as follows:

$$\sum_p |F_{p,j}(\tau)| \leqslant E_{14} \sum_p \sum_{l=1}^{\infty} \frac{1}{l^\gamma}\, \chi^{(l)}_{p,j}(\tau). \tag{4.44}$$

By means of (3.31) and (3.6), taking the supremum over μ, we can state

$$\sum_{j\in\tilde{a}} \sum_p \chi^{(l)}_{j,p}(\tau) \leqslant E_{15}\, l^3\, R(t)^3, \tag{4.45}$$

and by (4.32) it follows that

$$\sum_k 2^{-k} \sum_{j\in a_k} |v_j(s-\Delta)|^2 \leqslant E_{16}\, R(t)^3, \tag{4.46}$$

hence, combining these two relations and using the definition (4.4) of Δ, we get

$$\sum_k |a_k| 2^k \leqslant E_{17}\, R(t)^{14/6}\, \Delta. \tag{4.47}$$

It follows from (4.36), (4.41) and (4.47) that

$$\sum_k \left| \int_{s-\Delta}^{s} d\tau \sum_{j\in a_k} v_j \cdot F_{i,j} \right| \leqslant E_{18} R(t)^2\, \Delta, \tag{4.48}$$

since the sum over k of the second term of (4.36) can be easily bounded by:

$$\begin{aligned} R(t)^{4/6} \sum_k 2^k \sum_{j\in a_k} \frac{1}{\left[R(t)^{1/4}\right]^\gamma} \int_{s-\Delta}^{s} d\tau \ &\leqslant E_{19}\, R(t)^{4/6}\, R(t)^{14/6} \Delta^2\, R(t)^{-\gamma/4} \\ &\leqslant E_{19} \Delta\, R(t)^2. \end{aligned} \tag{4.49}$$

It remains to estimate the last contribution, namely that associated to the set of indices a_0. We have

$$\left| \int_{s-\Delta}^{s} d\tau \sum_{j\in a_0} v_j \cdot F_{i,j} \right| \leqslant E_{20} \sum_{h=0}^{H-1} R(t)^{4/6} \int_{s_h}^{s_{h+1}} d\tau \sum_{k=1}^{k_{\max}} \frac{1}{(kr)^\gamma}\, N^{(k)}(\tau) \tag{4.50}$$

with

$$N^{(k)}(\tau)=\sum_{j\in a_0}\chi(\,|q_i(\tau)-q_j(\tau)|<kr\,)$$

and where $J=[s-\Delta,s]$ has been decomposed into H identical intervals:

$$J=\bigcup_{h=0}^{H-1}[s_h,s_{h+1}] \tag{4.51}$$

with $s_H=s$, $s_0=s-\Delta$, and $|s_{h+1}-s_h|=\delta\in[\frac{1}{2\bar{A}\,R(t)},\frac{1}{\bar{A}\,R(t)}]$.

Moreover $k_{\max}$ is such that

$$k_{\max}=\left[R(t)^{1/4}/r\right]+1, \tag{4.52}$$

(such a choice for the maximum value of k will be clear later).

Since $|v_j(\tau)|\leqslant R(t)^{4/6}$, the maximal displacement of a particle belonging to the set a_0 is less than 1, in the time interval J. Moreover, defining

$$N_h^{(k)}=\sum_{j\in a_0}\chi(\inf_{\tau\in(s_h,s_{h+1})}|q_i(\tau)-q_j(s_0)|<kr+1) \tag{4.53}$$

for $\tau\in(s_h,s_{h+1})$, we get $N^{(k)}(\tau)\leqslant N_h^{(k)}$.

Then for (4.50) we have:

$$\begin{aligned}\left|\int_{s-\Delta}^{s}d\tau\sum_{j\in a_0}v_j\cdot F_{i,j}\right| &\leqslant E_{20}\,R(t)^{4/6}\,\delta\sum_{k=1}^{k_{max}}\frac{1}{(kr)^{\gamma}}\sum_{h=0}^{H-1}N_h^{(k)}\\ &\leqslant E_{20}\,R(t)^{4/6}\sqrt{H}\,\delta\sum_{k=1}^{k_{max}}\frac{1}{(kr)^{\gamma}}\left(\sum_{h=0}^{H-1}\left(N_h^{(k)}\right)^2\right)^{1/2}.\end{aligned} \tag{4.54}$$

Let us define

$$\mathcal{T}_h^k=\{y\in\mathbb{R}^3:\inf_{\tau\in(s_h,s_{h+1})}|q_i(\tau)-y|<kr+1\} \tag{4.55}$$

and

$$E(\mathcal{T}_h^k)=\sum_{l<j}\phi(q_l(s_0)-q_j(s_0))+b\,N(X^n(s_0),\mathcal{T}_h^k), \tag{4.56}$$

where the sum is restricted to the pairs of particles in $\mathcal{T}_h^k$ and E is a positive quantity because $b > B$. Let us note that $N(X^n(s_0), \mathcal{T}_h^k) \leqslant N_h^{(k)}$.

We want to estimate now the sum in (4.54)

$$\sum_{h=0}^{H-1} \left(N_h^{(k)}\right)^2. \tag{4.57}$$

If the sets $\mathcal{T}_h^k$ were all disjoint, then, defining

$$\mathcal{T}^k = \bigcup_h \mathcal{T}_h^k, \tag{4.58}$$

by superstability we would simply have

$$E(\mathcal{T}^k) \geqslant A \sum_{i \in \mathbb{Z}^3 \cap \mathcal{T}^k} n_{\Delta_i}^2 \geqslant A \sum_h \sum_{i \in \mathbb{Z}^3 \cap \mathcal{T}_h^k} n_{\Delta_i}^2 \geqslant E_{21} \sum_{h=0}^{H-1} \frac{\left(N_h^{(k)}\right)^2}{|\mathcal{T}_h^k|}. \tag{4.59}$$

We anticipate two results that will be proved afterwards: the first regarding $|\mathcal{T}_h^k|$

$$|\mathcal{T}_h^k| \leqslant E_{22}\, k^3, \tag{4.60}$$

the other dealing with the fact that a set $\mathcal{T}_h^k$ has a non empty intersection with no more than $(8+4rk)$ other sets (we consider k fixed).

In this way (4.59) becomes

$$E(\mathcal{T}^k) \geqslant \frac{E_{23}}{k^4} \sum_{h=0}^{H-1} \left(N_h^{(k)}\right)^2. \tag{4.61}$$

Putting the previous relation into (4.54), we can write

$$\left| \int_{s-\Delta}^{s} d\tau \sum_{j \in a_0} v_j \cdot F_{i,j} \right| \leqslant E_{24} R(t)^{4/6} \sqrt{H}\, \delta \sum_{k=1}^{k_{\max}} \frac{1}{(k)^{\gamma-2}}\; E(\mathcal{T}^k)^{1/2}. \tag{4.62}$$

By the bound on the maximal velocity of the ith particle

$$|v_i(\tau)| \leqslant \bar{A}\, R(t) + D_{21} R(t)^{3/2}\, \Delta \leqslant \frac{3}{2} \bar{A}\, R(t), \tag{4.63}$$

we get

$$\mathcal{T}^k \subset B(q_i(\tau), 2 + kr + R(t)^{1/3}) \tag{4.64}$$

with τ belonging to the interval $[s - \Delta, s]$ (for the proof see below).

Therefore

$$\begin{aligned} E(\mathcal{T}^k) &\leqslant \sum_{\substack{l<j: \\ q_l, q_j \in \mathcal{T}^k}} |\phi(q_l(s_0) - q_j(s_0))| + b\, N(X^n(s_0), \mathcal{T}^k) \\ &\leqslant \sum_{\substack{l<j: \\ q_l, q_j \in B}} |\phi_{l,j}| + b\, N(B) \leqslant E_{25} \sup_{\mu} \sum_{l} f_l^{\mu,R} \left(\sum_j |\phi_{l,j}| + b \right) \\ &\leqslant E_{26} \sup_{\mu} W(X^n(s_0); \mu, R(s_0)) \leqslant E_{27} R(t)^3, \end{aligned} \tag{4.65}$$

where in the fourth inequality we have used an estimate like the one given in (B.1) and Lemma 3.1, while in the last inequality we have used Lemma 3.2. Putting the last relation into (4.62) we get:

$$\left| \int_{s-\Delta}^{s} d\tau \sum_{j \in a_0} v_j \cdot F_{i,j} \right| \leqslant E_{28} \Delta\, R(t)^2, \tag{4.66}$$

since $\sqrt{H} = (\Delta/\delta)^{1/2} \leqslant \sqrt{2\zeta \bar{A}} R^{1/6}$.

It remains to prove that a fixed set $\mathcal{T}_h^k$ has a non empty intersection with no more than $(8 + 4rk)$ other sets, that $|\mathcal{T}_h^k| \leqslant E_{22}\, k^3$, and (4.64) (we will see that these three statements are consequences of the inclusion (4.69)). For a given h, let $e = \frac{v_i(s_{h+1})}{|v_i(s_{h+1})|}$ and $\xi(\tau) = (q_i(\tau) - q_i(s_{h+1})) \cdot e$. Then

$$\xi(\tau) = |v_i(s_{h+1})|\, (\tau - s_{h+1}) + \int_{s_{h+1}}^{\tau} d\sigma\, (\tau - \sigma)\, F_i(\sigma) \cdot e, \tag{4.67}$$

hence

$$
\begin{aligned}
|\xi(\tau)| &\geqslant |v_i(s_{h+1})|\,(\tau - s_{h+1}) - \frac{|\tau - s_{h+1}|^2}{2}\, D_{21}\, R(t)^{3/2} \\
&\geqslant |\tau - s_{h+1}|\,(\bar{A}R(t) - D_{21}R(t)^{3/2}R(t)^{-4/6}) \\
&\geqslant |\tau - s_{h+1}|\frac{\bar{A}R(t)}{2}
\end{aligned} \tag{4.68}
$$

for n large enough. On the other hand from (4.63) it follows that

$$
\mathcal{T}_h^k \subset B(q_i(s_{h+1}); \frac{3}{2}\bar{A}R(t)\delta + kr) \subset B(q_i(s_{h+1}); 2 + kr). \tag{4.69}
$$

Let us choose $|\tau - s_{h+1}| > (8+4rk)\delta$, with $(8+4rk)\delta \ll \Delta$ (such a condition guarantees us to remain in $[s-\Delta, s]$), that is $k \leqslant k_{\max} \ll R(t)^{1/3}$; from this last condition, the choice (4.52) previously done of taking $k_{\max} \sim R(t)^{1/4}$ is clear. Now, from (4.68), we have that $|\xi(\tau)| > 2 + kr$, and for this reason, after the time τ, q_i will not enter anymore into the ball $B(q_i(s_{h+1}); 2+kr)$, in such a way that $\mathcal{T}_h^k$ will have a non empty intersection with no more than $(8+4rk)$ other different $\mathcal{T}_y^k$'s.

The bound on $|\mathcal{T}_h^k|$ and the inclusion (4.64) are straightforward consequences of (4.69). ▮

We have now all the results necessary to prove the main theorem of this work.

5. PROOF OF THEOREM 2.2

Let us define the quantity

$$
\delta_i(n,t) = \left| q_i^n(t) - q_i^{n-1}(t) \right|. \tag{5.1}
$$

From the equations of motion in integral form we have:

$$
q_i^n(t) = q_i(0) + v_i(0)\,t + \int_0^t ds(t-s) \sum_{j:j\neq i} F\left(q_i^n(s) - q_j^n(s)\right). \tag{5.2}
$$

From (5.1) and (5.2) it follows that, for any $i \in I_{n-1}$,

$$
\delta_i(n,t) \leqslant \int_0^t ds(t-s)\left| \sum_{j:j\neq i} \left\{ \nabla\phi(q_i^n(s) - q_j^n(s)) - \nabla\phi(q_i^{n-1}(s) - q_j^{n-1}(s)) \right\} \right| \tag{5.3}
$$

and, because of the long-range of the interaction, it is useful to split up the last sum in the following way. Let

$$\min\left\{\left|q_i^{n-1}(s)-q_j^{n-1}(s)\right|,\ \left|q_i^n(s)-q_j^n(s)\right|\right\}=m_{ij}^n(s) \tag{5.4}$$

and, fixing a particle i, consider the following sets of indices:

$$\begin{aligned}\mathcal{A}_i^n(s,k) &= \left\{j\neq i:(k-1)\varphi(n)\leqslant m_{ij}^n(s)\leqslant k\varphi(n)\right\},\\ \tilde{\mathcal{A}}_i^n(s) &= \left\{j\neq i:m_{ij}^n(s)\geqslant k_{\max}\varphi(n)\right\}\end{aligned}$$

where $\varphi(n)=\psi_\xi(n)^{3/2}$ (ψ_ξ has been defined in (2.8)), $k=1,2,\ldots,k_{\max}$ and $k_{\max}=\left[n^{3/4}/\varphi(n)\right]$. We can write, using the property (2.5) of the interaction:

$$\begin{aligned}&\left|\sum_{j:j\neq i}\left\{\nabla\phi(q_i^n(s)-q_j^n(s))-\nabla\phi(q_i^{n-1}(s)-q_j^{n-1}(s))\right\}\right|\\ &\quad\leqslant L_1\sum_{j\in\mathcal{A}_i^n(s,1)}\left(\delta_i(n,s)+\delta_j(n,s)\right)\\ &\qquad+L_1\sum_{k=2}^{k_{\max}}\frac{1}{\left((k-1)\varphi(n)\right)^{\gamma+2}}\sum_{j\in\mathcal{A}_i^n(s,k)}\left(\delta_i(n,s)+\delta_j(n,s)\right)\\ &\qquad+L_1\frac{1}{\left(k_{\max}\varphi(n)\right)^{\gamma+2}}\sum_{j\in\tilde{\mathcal{A}}_i^n(s)}\left|q_i^n(s)-q_j^n(s)-q_i^{n-1}(s)+q_j^{n-1}(s)\right|.\end{aligned} \tag{5.5}$$

Defining

$$d_n(t)=\sup_{s\in[0,t]}\sup_{i\in I_n}\left|q_i^n(s)-q_i(0)\right|, \tag{5.6}$$

from the bound

$$V^n(t)\leqslant L_2\varphi(n), \tag{5.7}$$

(it is a consequence of (4.1), (4.2) and of Gronwall's lemma) we get, for $t\leqslant T$: $d_n(t)\leqslant L_3\varphi(n)$, where $L_3=L_2T$.

Hence, putting

$$p^{(k)}(n,t) = k\varphi(n) + L_3\varphi(n), \tag{5.8}$$

the number of particles contained in $\mathcal{A}_i^n(s,k)$ is bounded by the number of particles that, at the initial time, were in a ball of radius $p^{(k)}(n,t)$, and therefore, according to the definition (2.7), it is bounded by the quantity:

$$g^{(k)}(n,t) = Q_\xi(X)\left(p^{(k)}(n,t)\right)^3 \leqslant L_4 k^3 \varphi(n)^3. \tag{5.9}$$

For the same reason, the number of particles belonging to $\tilde{\mathcal{A}}_i^n(s)$ is bounded by $L_5 Q_\xi(X) n^3$, so the last term in (5.5) is bounded by

$$L_5 \frac{Q_\xi(X)\, n^4}{\left(n^{3/4}\right)^{\gamma+2}}. \tag{5.10}$$

We define

$$u_k(n,t) = \sup_{i \in I_k} \delta_i(n,t) \tag{5.11}$$

and we fix an integer $k_0 \ll n$. Putting

$$k_1 = \left[k_0 + p^{(k_{\max})}(n,t)\right], \tag{5.12}$$

we can bound the r.h.s. of (5.5) in the following way (using (5.8), (5.9), (5.10)):

$$(5.5) \leqslant L_1 \left(L_4 \varphi(n)^3 + \sum_{k \geqslant 2} \frac{L_4\, k^3 \varphi(n)^3}{\left((k-1)\varphi(n)\right)^{\gamma+2}} \right) u_{k_1}(n,s) + \frac{L_5\, Q_\xi(X)\, n^4}{\left(n^{3/4}\right)^{\gamma+2}}. \tag{5.13}$$

Hence by (5.3), (5.13), we get:

$$u_{k_0}(n,t) \leqslant L_6 \varphi(n)^3 \int_0^t ds\, (t-s)\, u_{k_1}(n,s) + \frac{L_7}{n^{(3/4)\gamma - 5/2}}. \tag{5.14}$$

We iterate now (5.14) m times, where m is

$$m = \left[\frac{n - k_0}{p^{(k_{\max})}(n,t)} \right]. \tag{5.15}$$

hence $u_m(n,t) \leqslant L_3 \varphi(n)$, we have

$$\begin{aligned} u_{k_0}(n,t) &\leqslant (L_8 \varphi(n))^{3m+1} \frac{t^{2m}}{(2m)!} + \frac{L_7}{n^{(3/4)\gamma - 5/2}} \sum_{h=1}^{m} \frac{\left(\varphi(n)^3\right)^h t^{2h}}{(2h)!} \\ &\leqslant (L_8 \varphi(n))^{3m+1} \frac{t^{2m}}{(2m)!} + \frac{L_7}{n^{(3/4)\gamma - 5/2}} \exp\left(\varphi(n)^{3/2} t\right). \end{aligned} \tag{5.16}$$

By the choice (5.15), using Stirling formula, since $\varphi(n)^{3/2} < L_9 (\log n)^{(9/4)\xi}$, where $\xi < 4/9$, and since $\gamma > 7$, it follows that $u_{k_0}(n,t)$ converges summably to zero as $n \to \infty$.

For what concernes the velocities we have:

$$\left| v_i^n(t) - v_i^{n-1}(t) \right| \leqslant \int_0^t ds \left| \sum_{j: j \neq i} F\left(q_i^n(s) - q_j^n(s)\right) - F\left(q_i^{n-1}(s) - q_j^{n-1}(s)\right) \right| \tag{5.17}$$

and we can bound the right hand side of (5.17) by the same estimates used to bound (5.5). In this way recalling (5.14) we obtain, for any $i \in I_{k_0}$:

$$\left| v_i^n(t) - v_i^{n-1}(t) \right| \leqslant L_6 \varphi(n)^3 \int_0^t ds\ u_{k_1}(n,s) + \frac{L_7}{n^{(3/4)\gamma - 5/2}}, \tag{5.18}$$

where for $u_{k_1}(n,s)$ it holds (5.16) replacing m with $m-1$:

$$u_{k_1}(n,t) \leqslant (L_8 \varphi(n))^{3(m-1)+1} \frac{t^{2(m-1)}}{(2(m-1))!} + \frac{L_7}{n^{(3/4)\gamma - 5/2}} \exp\left(\varphi(n)^{3/2} t\right). \tag{5.19}$$

Substituting (5.19) into (5.18) we have

$$\begin{aligned} \left| v_i^n(t) - v_i^{n-1}(t) \right| &\leqslant L_6 \varphi(n)^3 \\ &\times \int_0^t ds \left((L_8 \varphi(n))^{3(m-1)+1} \frac{s^{2(m-1)}}{(2(m-1))!} + \frac{L_7}{n^{(3/4)\gamma - 5/2}} \exp(\varphi(n)^{3/2} s) \right) \\ &+ \frac{L_7}{n^{(3/4)\gamma - 5/2}} \end{aligned} \tag{5.20}$$

from which it follows that $\left|v_i^n(t)-v_i^{n-1}(t)\right|$ converges summably to zero as $n\to\infty$.

To prove that the limit solution belongs to (2.15) for any time $0\leqslant t\leqslant T$, with T arbitrary but a priori fixed, let us fix $i\in\mathbb{N}$ and choose k_0 such that $k_0-1\leqslant |q_i|\leqslant k_0$. We choose n^* of the form

$$n^*=\left[k_0^2+L_{10}\right], \tag{5.21}$$

in such a way that we have a uniform convergence of $\sum_{n\geqslant n^*} u_{k_0}(n,t)$ with respect to k_0 (as it appears evident from (5.15)). Now we have:

$$|v_i(t)-v_i^{n^*}(t)|\leqslant \sum_{n\geqslant n^*} |v_i^n(t)-v_i^{n-1}(t)|, \tag{5.22}$$

hence by (5.20) and by the choice made for n^*, the right hand side of (5.22) is bounded by a constant indipendent from k_0:

$$|v_i(t)|\leqslant |v_i^{n^*}(t)|+L_{11}. \tag{5.23}$$

Thus from (5.7) it follows

$$\begin{aligned}|v_i^{n^*}(t)| &\leqslant L_{12}\big(\log(e+n^*)\big)^{\frac{3}{2}\xi}\leqslant L_{13}\big(\log(e+k_0)\big)^{\frac{3}{2}\xi}\\ &\leqslant L_{14}\big(\log(e+|q_i|)\big)^{\frac{3}{2}\xi}=L_{14}\psi_\xi^{3/2}(|q_i|),\end{aligned} \tag{5.24}$$

so that, from (5.24) and (5.23), it follows

$$|v_i(t)|\leqslant L_{15}\psi_\xi^{3/2}(|q_i|). \tag{5.25}$$

We want to prove now that, if $X\in\mathcal{X}_\xi$, then $X(t)\in\mathcal{X}_{\frac{3}{2}\xi}$.
Given $\mu\in\mathbb{R}^3$ and $R>(\log(e+|\mu|))^{\frac{3}{2}\xi}$ let

$$n_0=\left[L_{16}\exp\left(2\,R^{\frac{2}{3\xi}}\right)\right]. \tag{5.26}$$

Clearly $(\log(e+n_0))^{\frac{3}{2}\xi}\geqslant R$ so that, by Lemma 3.2 and from the relation

$$Q(X;\mu,R)\leqslant \widetilde{L}\,W(X;\mu,R),$$

(see (C.1)), we have

$$Q(X^{n_0}(t);\mu,R)\leqslant \widetilde{L}\,W(X^{n_0}(t);\mu,2R(n_0,t))\leqslant L_{17}R^3(n_0,t)$$
$$\leqslant L_{18}(\log(e+n_0))^{\frac{9}{2}\xi}\leqslant L_{19}\left(R^{\frac{2}{3\xi}}\right)^{\frac{9\xi}{2}}\leqslant L_{20}R^3. \quad (5.27)$$

On the other hand

$$Q(X(t);\mu,R)\leqslant Q(X^{n_0}(t);\mu,R) + \sum_{n>n_0}\left|Q(X^n(t);\mu,R)-Q(X^{n-1}(t);\mu,R)\right| \quad (5.28)$$

and the sum on the r.h.s. of (5.28), by the choice (5.26) of n_0 (which in particular implies that $n_0>|\mu|$), converges uniformly with respect to $\mu\in\mathbb{R}^3$ and $R>(\log(e+|\mu|))^{\frac{3}{2}\xi}$, so it is bounded by a constant independent from μ and R.

Notice that the following inequalities hold

$$(\log(e+n_0))^{\frac{3}{2}\xi}\geqslant R\geqslant(\log(e+|\mu|))^{\frac{3}{2}\xi}, \quad (5.29)$$

in order that, combining (5.27) and (5.28), taking the supremum over $\mu\in\mathbb{R}^3$ and over $R>(\log(e+|\mu|))^{\frac{3}{2}\xi}$, we obtain that $X(t)\in\mathcal{X}_{\frac{3}{2}\xi}$.

We want to underline that we cannot say that the solution surely exits from $\mathcal{X}_\xi$, we have only proved that the maximal set of existence for $X(t)$ is $\mathcal{X}_{\frac{3}{2}\xi}\supset\mathcal{X}_\xi$.

For what concerns the uniqueness of the solution, let us assume that there is a solution $\{q_i^\star,v_i^\star\}$ different from the one obtained as the limit of the partial dynamics and deduce a contradiction. In the space defined by (2.14) and (2.15) it can be easily proved that the difference $|q_i^n-q_i^\star|$ converges to zero as $n\to\infty$ by an iterative method identical to the one just used, in particular we need the restriction over the velocities provided by (2.15) in order to make the iterative method work. This last condition on the velocities is imposed by the long-range character of the interaction, which gives origin to a term like the last present in (5.16).

We want to point out that the restriction (2.15) is a requirement imposed to prove the uniqueness of the solution. In particular we need a velocity bound (better than the one following by energy conservation) for the non-limit (hypothetical) solution $\{q_i^\star,v_i^\star\}$, necessary to make the iterative method work. Nevertheless we remark that we have proved that the limit solution, $\lim_{n\to\infty}\{q_i^n,v_i^n\}$, belongs directly to (2.15). ∎

The proof of Theorem 2.1, dealing with the short-range interaction, is analogous to Theorem 2.2's, with obvious simplifications.

APPENDIX A

Proof of Lemma 3.1. We make a partition of the physical space with large cubes of side mr and we divide the interaction into a short-range and a long-range one. The last one can be handled using Proposition 2.1. Concerning the short-range interaction we choose the parameter α in the definition of the weight-function f (see (2.24)) so small in such a way that, in a cube, f is constant, and then, neglecting the interaction with the other cubes, W is superstable. Of course the interaction between different cubes exists, but it gives a surface effect, and it becomes negligible with respect to a volume effect, as m is very large.

Let us define the set $\Gamma^l_u(r)$ in the following way

$$\Gamma^l_u(r) \equiv \{x \in \mathbb{R}^3 : u^{(i)} + l^{(i)}mr \leqslant x^{(i)} < u^{(i)} + (l^{(i)}+1)mr, \\ u \in \mathbb{R}^3;\ m \in \mathbb{N},\ l \in \mathbb{Z}^3\},$$

where r is the parameter appearing in Proposition 2.1.

From this definition, it follows that $|x-y| \leqslant \sqrt{3}mr, \quad \forall x, y \in \Gamma^l_u(r)$, then, by the properties of the weight-function

$$f(|y-\mu|, R) \leqslant f(|x-\mu|, R)\left(1+\alpha\sqrt{3}mr\right)^{\lambda}. \tag{A.1}$$

We define also the following quantities:

$$\widehat{f}^{\mu,R}_{u,l} \equiv \inf_{i\in\mathbb{N}, q_i\in\Gamma^l_u} f^{\mu,R}_i, \tag{A.2}$$

$$P^l_u(r) \equiv \{(i,j) \in \mathbb{N}\otimes\mathbb{N} : i > j,\ q_i \in \Gamma^l_u,\ q_j \in \Gamma^l_u,\ |q_i - q_j| < r\}, \tag{A.3}$$

$$T_u(r) \equiv \bigcup_{l\in\mathbb{Z}^3} P^l_u(r), \tag{A.4}$$

$$M \equiv \sup_{x\in\mathbb{R}^3} |\phi(x)|, \tag{A.5}$$

$$V(r) \equiv \{(i,j) \in \mathbb{N}\otimes\mathbb{N} : i > j,\ |q_i - q_j| < r\}. \tag{A.6}$$

As it follows from its definition, $V(r)$ is the set of all the pairs of particles with relative distance smaller than r, while in T_u there are no pairs with particles in two adjoining Γ^l_u.

Let ϵ be a real positive number such that

$$\epsilon > \sqrt{3}\alpha mr, \tag{A.7}$$

then for each x and y in Γ_u^l we have:

$$f(|y-\mu|, R) \leqslant (1+\epsilon)^\lambda f(|x-\mu|, R). \tag{A.8}$$

Since the potential can be decomposed into $\phi = \phi^{(1)} + \phi^{(2)}$ (see (2.17)), the mollified energy becomes

$$W(X; \mu, R) \equiv W^{(1)}(X; \mu, R) + W^{(2)}(X; \mu, R), \tag{A.9}$$

where

$$W^{(1)}(X; \mu, R) \equiv \sum_{i\in\mathbb{N}} f_i^{\mu,R} \Big(\frac{v_i^2}{2} + \frac{1}{2} \sum_{j:j\neq i} \phi_{i,j}^{(1)} + b \Big), \tag{A.10}$$

$$W^{(2)}(X; \mu, R) \equiv \sum_{i\in\mathbb{N}} f_i^{\mu,R} \frac{1}{2} \sum_{j:j\neq i} \phi_{i,j}^{(2)}. \tag{A.11}$$

Let us estimate now the second term $W^{(2)}$. For r large enough we have:

$$\begin{aligned}
|W^{(2)}| &\leqslant \widetilde{D}_1 \left(1+\sqrt{3}\,\alpha\right)^\lambda \sum_{i\in\mathbb{Z}^3} f(|i-\mu|, R)\, n_{\Delta_i} \\
&\quad \times \sum_{j\in\mathbb{Z}^3} n_{\Delta_j} \frac{\chi(|i-j|>r-2)}{(|i-j|-\sqrt{3})^\gamma} \\
&\leqslant \widetilde{D}_2 \sum_{i\in\mathbb{Z}^3} \sum_{\substack{j\in\mathbb{Z}^3:\\ |i-j|>r-2}} f(|i-\mu|, R)\, n_{\Delta_i} n_{\Delta_j} \frac{1}{|i-j|^\gamma} \\
&\leqslant \widetilde{D}_3 \sum_{i\in\mathbb{Z}^3} \sum_{\substack{j\in\mathbb{Z}^3:\\ |i-j|>r-2}} f(|i-\mu|, R) \left(n_{\Delta_i}^2 + n_{\Delta_j}^2\right) \frac{1}{|i-j|^\gamma} \\
&\leqslant \widetilde{D}_4 \Bigg\{ \sum_{i\in\mathbb{Z}^3} \sum_{\substack{j\in\mathbb{Z}^3:\\ |i-j|>r-2}} f(|i-\mu|, R)\, n_{\Delta_i}^2 \frac{1}{|i-j|^\gamma} \\
&\quad + \sum_{\substack{i,j\in\mathbb{Z}^3:\\ |i-j|>r-2}} f(|j-\mu|, R)\, n_{\Delta_j}^2 \frac{1}{|i-j|^\gamma} (1+|i-j|)^\lambda \Bigg\}
\end{aligned}$$

$$\leqslant \widetilde{D}_5 \sum_{i\in\mathbb{Z}^3} f(|i-\mu|, R)\, n_{\Delta_i}^2 \times \sum_{k=r}^{\infty} \sum_{j\in\mathbb{Z}^3} \chi(k\leqslant |j| < (k+1))\, \frac{(1+k+1)^{\lambda}}{k^{\gamma}}$$

$$\leqslant \widetilde{D}_6 \sum_{i\in\mathbb{Z}^3} f(|i-\mu|, R)\, n_{\Delta_i}^2 \sum_{k=r}^{\infty} \frac{1}{k^{\gamma-2-\lambda}}$$

$$\leqslant \widetilde{D}_7(r) \sum_{i\in\mathbb{Z}^3} f(|i-\mu|, R)\, n_{\Delta_i}^2, \tag{A.12}$$

with $\widetilde{D}_7(r)$ such that:

$$\lim_{r\to+\infty} \widetilde{D}_7(r) = 0, \tag{A.13}$$

as

$$\gamma > 3+\lambda. \tag{A.14}$$

Therefore $\exists\, r_1 > 0 : \forall r > r_1 \Rightarrow \widetilde{D}_7(r) \leqslant \frac{1}{4}A$, hence

$$W^{(2)} \geqslant -\frac{1}{4} A \sum_{i\in\mathbb{Z}^3} f(|i-\mu|, R)\, n_{\Delta_i}^2 \tag{A.15}$$

for any $r > r_1$.

Now it remains to examine the first term $W^{(1)}$.
If we define the quantity

$$E(X;\mu,\Gamma_u^l) = \sum_{(i,j)\in P_u^l} f_i^{\mu,R} \phi_{i,j}^{(1)}, \tag{A.16}$$

by the superstability of $\phi^{(1)}$ we have

$$\begin{aligned} E(X;\mu,\Gamma_u^l) &= \sum_{(i,j)\in P_u^l} (f_i^{\mu,R} - \widehat{f}_{u,l}^{\mu,R})\phi_{i,j}^{(1)} + \widehat{f}_{u,l}^{\mu,R} \sum_{(i,j)\in P_u^l} \phi_{i,j}^{(1)} \\ &\geqslant -M((1+\epsilon)^{\lambda}-1) \sum_{(i,j)\in P_u^l} \widehat{f}_{u,l}^{\mu,R} - B\widehat{f}_{u,l}^{\mu,R} \sum_{k\in\mathbb{Z}_u^3} n_{\Delta_k} \\ &\quad + \frac{3}{4} A \widehat{f}_{u,l}^{\mu,R} \sum_{k\in\mathbb{Z}_u^3} n_{\Delta_k}^2 \end{aligned}$$

and from the following definition

$$\mathbb{Z}_u^3 \equiv \mathbb{Z}^3 \cap \Gamma_u^l \tag{A.17}$$

we get

$$E(X;\mu,\Gamma_u^l) \geqslant -B\,\widehat{f}_{u,l}^{\mu,R} \sum_{k\in\mathbb{Z}_u^3} n_{\Delta_k} + \frac{3A}{4(1+\epsilon)^\lambda}\sum_{k\in\mathbb{Z}_u^3} f(|k-\mu|,R)\,n_{\Delta_k}^2$$
$$-\frac{M}{2}((1+\epsilon)^\lambda-1)\sum_{(i,j)\in P_u^l}(f_i^{\mu,R}+f_j^{\mu,R}). \tag{A.18}$$

Choosing $z=u$, where $z\in\Gamma_0^0\cap r\mathbb{Z}^3$, we have $\cup_{l\in\mathbb{Z}^3}\Gamma_z^l=\mathbb{R}^3$, and to each z it is associated a partition $\mathcal{P}_z$ of the space.

For a fixed partition, considering the definition (2.23), summing (A.16) over the sets $\Gamma_z^l\in\mathcal{P}_z$ and taking into account all the contributions of the pairs not belonging to a set of the partition, we finally obtain a lower bound for the mollified energy. Indeed choosing $b>B$, we have:

$$W^{(1)}(X;\mu,R) \geqslant \sum_{l\in\mathbb{Z}^3} E(X;\mu,\Gamma_z^l) + b\sum_{l\in\mathbb{Z}^3}\widehat{f}_{z,l}^{\mu,R}\,n_{\Delta_l}$$
$$-\frac{M}{2}\sum_{(i,j)\notin T_z}(f_i^{\mu,R}+f_j^{\mu,R})$$
$$\geqslant \frac{3A}{4(1+\epsilon)^\lambda}\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\,n_{\Delta_k}^2 - M\sum_{(i,j)\notin T_z}(f_i^{\mu,R}+f_j^{\mu,R})$$
$$-\frac{M}{2}((1+\epsilon)^\lambda-1)\sum_{(i,j)\in T_z}(f_i^{\mu,R}+f_j^{\mu,R})$$
$$\geqslant \frac{3A}{4(1+\epsilon)^\lambda}\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\,n_{\Delta_k}^2 - M\sum_{(i,j)\notin T_z}(f_i^{\mu,R}+f_j^{\mu,R})$$
$$-\frac{M}{2}((1+\epsilon)^\lambda-1)\sum_{(i,j)\in V}(f_i^{\mu,R}+f_j^{\mu,R}). \tag{A.19}$$

If we sum over z, the term in the left hand side is clearly independent of z. On the contrary, given a pair of particles (i,j), the number of z such that $(i,j)\in T_z$ is larger than $(m-2)^3$, thus the number of pairs of particles with a relative distance smaller than r, but such that they do not belong to T_z, is less than $m^3-(m-2)^3\leqslant 14\,m^2$.

In this way we obtain

$$m^3 W^{(1)}(X;\mu,R) \geqslant \frac{3A\,m^3}{4(1+\epsilon)^\lambda}\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\,n^2_{\Delta_k}$$
$$-14M\,m^2\sum_{(i,j)\in V}(f_i^{\mu,R}+f_j^{\mu,R})$$
$$-\frac{M\,m^3}{2}((1+\epsilon)^\lambda-1)\sum_{(i,j)\in V}(f_i^{\mu,R}+f_j^{\mu,R}). \tag{A.20}$$

Let us estimate now the last two terms of the sum:

$$\begin{aligned}\sum_{(i,j)\in V}(f_i^{\mu,R}+f_j^{\mu,R}) &\leqslant \sum_{i<j}(f_i^{\mu,R}+f_j^{\mu,R})\chi(|q_i-q_j|<r)\\ &\leqslant \widetilde{D}_8\sum_{i\in\mathbb{Z}^3}\sum_{\substack{l\in\mathbb{N}:\\ q_l\in\Delta_i}}\sum_{\substack{j\in\mathbb{Z}^3:\\ |i-j|<r+2}}\sum_{g\in\mathbb{N}:q_g\in\Delta_j} f(|i-\mu|,R)\\ &= \widetilde{D}_8\sum_{i\in\mathbb{Z}^3}\sum_{\substack{j\in\mathbb{Z}^3:\\ |i-j|<r+2}} f(|i-\mu|,R)\,n_{\Delta_i}n_{\Delta_j}.\end{aligned}$$

Obviously for a fixed j

$$\mathbf{Card}\{i\in\mathbb{Z}^3:|i-j|<r+2\}\leqslant\widetilde{D}_9\,r^3,$$

then, for r large enough:

$$\sum_{(i,j)\in V}(f_i^{\mu,R}+f_j^{\mu,R})\leqslant\widetilde{D}_{10}\,r^{3+\lambda}\sum_{i\in\mathbb{Z}^3}f(|i-\mu|,R)\,n^2_{\Delta_i}. \tag{A.21}$$

In conclusion the term $W^{(1)}$ is bounded by

$$W^{(1)}(X;\mu,R)\geqslant D(\epsilon,m,r)\sum_{i\in\mathbb{Z}^3}f(|i-\mu|,R)\,n^2_{\Delta_i}, \tag{A.22}$$

where

$$D(\epsilon,m,r)\equiv\left(\frac{3A}{4(1+\epsilon)^\lambda}-\frac{14\widetilde{D}_{10}M}{m}r^{3+\lambda}-\frac{M}{2}\widetilde{D}_{10}r^{3+\lambda}((1+\epsilon)^\lambda-1)\right). \tag{A.23}$$

Let r be such that $r > \max\{\bar{r}, r_1\}$, and m such that:

$$m \geqslant \bar{m} \equiv \frac{112\widetilde{D}_{10} M r^{3+\lambda}}{A}, \tag{A.24}$$

and let ϵ satisfy the following bound:

$$\epsilon \leqslant \min\left\{(3/2)^{\frac{1}{\lambda}} - 1, \left(\frac{A}{4M\widetilde{D}_{10} r^{3+\lambda}} + 1\right)^{\frac{1}{\lambda}} - 1\right\}, \tag{A.25}$$

so that we have

$$D(\epsilon, m, r) \geqslant \frac{1}{4} A. \tag{A.26}$$

Finally we fix α in such a way that $\alpha m r\sqrt{3} < \epsilon$, so the thesis immediately follows with $C_3 = 1/4A$.

Summing up, first we choose r so large that the tail term W_2 is small enough. Then, for a fixed r, m can be chosen in such a way that (A.24) holds, and ϵ small enough to satisfy (A.25). Finally, as we have fixed r, m, ϵ, from (A.7) the bound on α follows. ▮

APPENDIX B

Proof of Corollary 3.1. The only part which remains to prove is:

$$\sum_{i\in\mathbb{N}} f_i^{\mu,R} \left(\frac{1}{2} \sum_{\substack{j\in\mathbb{N}:\\ j\neq i}} \phi_{i,j} + b\right) \leqslant C_4 \sum_{k\in\mathbb{Z}^3} f(|k-\mu|, R)\, n_{\Delta_k}^2. \tag{B.1}$$

Let us define

$$\sum_{i\in\mathbb{N}} f_i^{\mu,R} \left(\frac{1}{2} \sum_{\substack{j\in\mathbb{N}:\\ j\neq i}} \phi_{i,j} + b\right) \equiv W^{(a)} + W^{(b)}, \tag{B.2}$$

where

$$W^{(a)} \equiv \sum_{i\in\mathbb{N}} f_i^{\mu,R} \left(\frac{1}{2} \sum_{\substack{j\in\mathbb{N}:\\ j\neq i}} \phi_{i,j}^{(1)} + b \right), \tag{B.3}$$

$$W^{(b)} \equiv \frac{1}{2} \sum_{i\in\mathbb{N}} f_i^{\mu,R} \sum_{\substack{j\in\mathbb{N}:\\ j\neq i}} \phi_{i,j}^{(2)}. \tag{B.4}$$

Using the third property of Proposition 2.2, the first term can be easily bounded by

$$\begin{aligned} W^{(a)} \leqslant (1+\alpha\sqrt{3})^{\lambda}\, \widetilde{E}_1 \sum_{l\in\mathbb{Z}^2} f(|l-\mu|, R)\, n_{\Delta_l} \\ + \frac{\|\phi^{(1)}\|_\infty}{2} \sum_{i\neq j} f_i^{\mu,R}\, \chi(|q_i - q_j| \leqslant r). \end{aligned} \tag{B.5}$$

Thus

$$W^{(a)} \leqslant \widetilde{E}_2 \sum_{l\in\mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l}^2 + \widetilde{E}_3 \sum_{i\neq j} f_i^{\mu,R} \chi(|q_j - q_j| \leqslant r). \tag{B.6}$$

Let us give an upper bound for the second term that we denote with $\widetilde{W}$:

$$\begin{aligned} \widetilde{W} &\equiv \sum_{i\neq j} f_i^{\mu,R} \chi(|q_j - q_j| \leqslant r) \\ &\leqslant \sum_{l,m\in\mathbb{Z}^3} \sum_{i\neq j} \chi_i(\Delta_l)\chi_j(\Delta_m)(1+\alpha\sqrt{3})^{\lambda} f(|l-\mu|, R)\, \chi(|l-m| \leqslant r+\sqrt{3}) \\ &\leqslant \widetilde{E}_4 \sum_{l\in\mathbb{Z}^3} \sum_{m\in\mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l} n_{\Delta_m} \chi(|l-m| \leqslant r+\sqrt{3}) \\ &\leqslant \frac{\widetilde{E}_4}{2} \sum_{l\in\mathbb{Z}^3} \sum_{m\in\mathbb{Z}^3} f(|l-\mu|, R)\, (n_{\Delta_l}^2 + n_{\Delta_m}^2) \chi(|l-m| \leqslant r+\sqrt{3}) \\ &\leqslant \frac{\widetilde{E}_4}{2} \sum_{l\in\mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l}^2 \sum_{m\in\mathbb{Z}^3} \chi(|l-m| \leqslant r+\sqrt{3}) \\ &\quad + \frac{\widetilde{E}_4}{2} \sum_{m\in\mathbb{Z}^3} f(|m-\mu|, R)\, n_{\Delta_m}^2 \sum_{l\in\mathbb{Z}^3} (1+\alpha(r+\sqrt{3}))^{\lambda} \chi(|l-m| \leqslant r+\sqrt{3}) \\ &\leqslant \widetilde{E}_5\, r^{3+\lambda} \sum_{l\in\mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l}^2 \leqslant \widetilde{E}_6 \sum_{l\in\mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l}^2, \end{aligned}$$

where we denote with $\chi_i(\Delta_l)$ the characteristic function of the set $\{i \in \mathbb{N} : q_i \in \Delta_l\}$ and with n_{Δ_l} the number of particles in the unit cube Δ_l with its center in l. Moreover we have used the fact that, for a fixed l, $\sum_{m\in\mathbb{Z}^3} \chi(|l-m| \leqslant r+\sqrt{3})$ is bounded by the cardinality of the set $\mathbb{Z}^3 \cap B(0, r+\sqrt{3})$.

Thus for $W^{(a)}$ we have

$$W^{(a)} \leqslant \widetilde{E}_7 \sum_{l\in\mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l}^2. \tag{B.7}$$

Let us give a similar estimate for $W^{(b)}$.

From the forth property of (2.17) we have:

$$\begin{aligned} W^{(b)} &\leqslant \widetilde{E}_8 \sum_{i\in\mathbb{N}} f_i^{\mu,R} \sum_{j\in\mathbb{N}} \chi(|q_j - q_j| \geqslant r) \frac{1}{|q_i - q_j|^\gamma} \\ &= \widetilde{E}_8 \sum_{k=1}^{\infty} \sum_{i,j} f_i^{\mu,R} \chi(kr \leqslant |q_i - q_j| < (k+1)r) \frac{1}{|q_i - q_j|^\gamma}, \end{aligned}$$

thus

$$\begin{aligned} W^{(b)} &\leqslant \widetilde{E}_8 \sum_{k=1}^{\infty} \frac{1}{(kr)^\gamma} \sum_{\substack{l,m\in\\ \mathbb{Z}^3}} \sum_{\substack{i,j\in\\ \mathbb{N}}} \chi_i(\Delta_l)\chi_j(\Delta_m) \\ &\quad \times (1+\alpha\sqrt{3})^\lambda f(|l-\mu|, R)\ \chi(kr - \sqrt{3} \leqslant |l-m| \leqslant (k+1)r + \sqrt{3}) \\ &\leqslant \widetilde{E}_9 \sum_{k=1}^{\infty} \frac{1}{(kr)^\gamma} \\ &\quad \times \sum_{\substack{l,m\in\\ \mathbb{Z}^3}} f(|l-\mu|, R)\, n_{\Delta_l} n_{\Delta_m} \chi(kr - \sqrt{3} \leqslant |l-m| \leqslant (k+1)r + \sqrt{3}) \\ &\leqslant \frac{\widetilde{E}_9}{2} \sum_{k=1}^{\infty} \frac{1}{(kr)^\gamma} \sum_{\substack{l,m\in\\ \mathbb{Z}^3}} \left(f(|l-\mu|, R) n_{\Delta_l}^2 + f(|m-\mu|, R)(1+\alpha|l-m|)^\lambda n_{\Delta_m}^2 \right) \\ &\quad \times \chi(kr - \sqrt{3} \leqslant |l-m| \leqslant (k+1)r + \sqrt{3}) \\ &\leqslant \widetilde{E}_{10} \sum_{k=1}^{\infty} \frac{k^2 r^3}{(kr)^\gamma} \sum_{l\in\mathbb{Z}^3} f(|l-\mu|, R)\, n_{\Delta_l}^2 \end{aligned}$$

$$
+\widetilde{E}_{11}\sum_{k=1}^{\infty}\frac{k^2(1+\alpha((k+1)r+\sqrt{3}))^{\lambda}}{(kr)^{\gamma}}\sum_{m\in\mathbb{Z}^3} f(|m-\mu|,R)\,n_{\Delta_m}^2
$$

$$
\leqslant \widetilde{E}_{12}\sum_{l\in\mathbb{Z}^3} f(|l-\mu|,R)\,n_{\Delta_l}^2, \tag{B.8}
$$

where in the last inequality the convergence of the series follows from the bound on γ.

Thus we have:

$$
\sum_{i\in\mathbb{N}} f_i^{\mu,R}\left(\frac{1}{2}\sum_{\substack{j\in\mathbb{N}:\\ j\neq i}}\phi_{i,j}+b\right)\leqslant C_4\sum_{l\in\mathbb{Z}^3} f(|l-\mu|,R)\,n_{\Delta_l}^2, \tag{B.9}
$$

and then the proof easily follows. ∎

APPENDIX C

Proof of Corollary 3.2. For the first inequality we prove a stronger bound: there exists a positive constant $\widetilde{L}$ such that:

$$
Q(X;\mu,R)\leqslant \widetilde{L}\,W(X;\mu,R). \tag{C.1}
$$

From definition (2.6) we can write:

$$
\begin{aligned}
Q(X;\mu,R) &= \sum_{i\in\mathbb{N}}\chi(|q_i-\mu|\leqslant R)\frac{v_i^2}{2}\\
&\quad+\sum_{i\in\mathbb{N}}\chi(|q_i-\mu|\leqslant R)\left(b+\frac{1}{2}\sum_{\substack{j\neq i:\\ q_j\in B(\mu,R)}}\phi_{i,j}^{(1)}\right)\\
&\quad+\frac{1}{2}\sum_{i\in\mathbb{N}}\chi(|q_i-\mu|\leqslant R)\sum_{\substack{j\neq i:\\ q_j\in B(\mu,R)}}\phi_{i,j}^{(2)}\\
&\equiv T+U^{(1)}+U^{(2)},
\end{aligned} \tag{C.2}
$$

where

$$
T\equiv\sum_{i\in\mathbb{N}}\chi(|q_i-\mu|\leqslant R)\frac{v_i^2}{2}
$$

$$U^{(1)} \equiv \sum_{i\in\mathbb{N}} \chi(|q_i-\mu|\leqslant R)\left(b+\frac{1}{2}\sum_{\substack{j\neq i:\\ q_j\in B(\mu,R)}} \phi^{(1)}_{i,j}\right),$$

$$U^{(2)} \equiv \frac{1}{2}\sum_{i\in\mathbb{N}} \chi(|q_i-\mu|\leqslant R)\sum_{\substack{j\neq i:\\ q_j\in B(\mu,R)}} \phi^{(2)}_{i,j}.$$

Because of the boundness of the weight-function $f_i^{\mu,R}$ and from the positivity of the interaction energy (3.2), we have

$$T\leqslant \widetilde{L}_1\sum_{i\in\mathbb{N}} f_i^{\mu,R}\frac{v_i^2}{2}\leqslant \widetilde{L}_1 W(X;\mu,R), \tag{C.3}$$

The second term can be easily bounded by

$$\begin{aligned}
U^{(1)}\leqslant\ & \widetilde{L}_1\sum_{i\in\mathbb{N}} f_i^{\mu,R}b+\frac{1}{2}\sum_{i\neq j}\chi(|q_i-\mu|\leqslant R)\chi(|q_j-\mu|\leqslant R)\ |\phi^{(1)}_{i,j}|\\
\leqslant\ & (1+\alpha\sqrt{3})^{\lambda}\,\widetilde{L}_1\sum_{l\in\mathbb{Z}^3} f(|l-\mu|,R)\,n_{\Delta_l}\\
&+\frac{\|\phi^{(1)}\|_\infty}{2}\sum_{i\neq j}\chi(|q_i-\mu|\leqslant R)\chi(|q_i-q_j|\leqslant r),
\end{aligned} \tag{C.4}$$

where for the first addendum we have used the third property of Proposition 2.2.

Thus

$$U^{(1)}\leqslant \widetilde{L}_2\sum_{l\in\mathbb{Z}^3} f(|l-\mu|,R)\,n^2_{\Delta_l}+\widetilde{L}_3\sum_{i\neq j} f_i^{\mu,R}\chi(|q_j-q_j|\leqslant r). \tag{C.5}$$

Let us give an upper bound for the second term that we denote with $\widetilde{U}$

$$\begin{aligned}
\widetilde{U}=&\sum_{i\neq j} f_i^{\mu,R}\chi(|q_j-q_j|\leqslant r)\leqslant \sum_{l,m\in\mathbb{Z}^3}\sum_{i\neq j}\chi_i(\Delta_l)\chi_j(\Delta_m)(1+\alpha\sqrt{3})^{\lambda}\\
&\times f(|l-\mu|,R)\,\chi(|l-m|\leqslant r+\sqrt{3})
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \widetilde{L}_4 \sum_{l\in\mathbb{Z}^3}\sum_{m\in\mathbb{Z}^3} f(|l-\mu|,R)\, n_{\Delta_l} n_{\Delta_m}\chi(|l-m|\leqslant r+\sqrt{3})\\
&\leqslant \frac{\widetilde{L}_4}{2}\sum_{l\in\mathbb{Z}^3}\sum_{m\in\mathbb{Z}^3} f(|l-\mu|,R)\,(n_{\Delta_l}^2+n_{\Delta_m}^2)\chi(|l-m|\leqslant r+\sqrt{3})\\
&\leqslant \frac{\widetilde{L}_4}{2}\sum_{l\in\mathbb{Z}^3} f(|l-\mu|,R)\, n_{\Delta_l}^2 \sum_{m\in\mathbb{Z}^3}\chi(|l-m|\leqslant r+\sqrt{3})\\
&\quad +\frac{\widetilde{L}_4}{2}\sum_{m\in\mathbb{Z}^3} f(|m-\mu|,R)\, n_{\Delta_m}^2 \sum_{l\in\mathbb{Z}^3}(1+\alpha(r+\sqrt{3}))^{\lambda}\chi(|l-m|\leqslant r+\sqrt{3})\\
&\leqslant \widetilde{L}_5 r^{3+\lambda}\sum_{l\in\mathbb{Z}^3} f(|l-\mu|,R)\, n_{\Delta_l}^2 \leqslant \widetilde{L}_6 \sum_{l\in\mathbb{Z}^3} f(|l-\mu|,R)\, n_{\Delta_l}^2,
\end{aligned}
$$

where we denote with $\chi_i(\Delta_l)$ the characteristic function of the set $\{i\in\mathbb{N}: q_i\in\Delta_l\}$ and with n_{Δ_l} the number of particles in the unit cube Δ_l with its center in l. Moreover we have used the fact that, for a fixed l, $\sum_{m\in\mathbb{Z}^3}\chi(|l-m|\leqslant r+\sqrt{3})$ is bounded by the cardinality of the set $\mathbb{Z}^3\cap B(0,r+\sqrt{3})$.

Thus for $U^{(1)}$ we have

$$
U^{(1)} \leqslant \widetilde{L}_7 \sum_{l\in\mathbb{Z}^3} f(|l-\mu|,R)\, n_{\Delta_l}^2. \tag{C.6}
$$

Let us give a similar estimate for $U^{(2)}$.

From the forth property of (2.17) we have

$$
\begin{aligned}
U^{(2)} &\leqslant G_1 \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \chi(|q_j-\mu|\leqslant R)\chi(|q_j-q_j|\geqslant r)\frac{1}{|q_i-q_j|^{\gamma}}\\
&\leqslant \widetilde{L}_8 \sum_{i\in\mathbb{N}} f_i^{\mu,R}\sum_{j\in\mathbb{N}}\chi(|q_j-q_j|\geqslant r)\frac{1}{|q_i-q_j|^{\gamma}}\\
&\leqslant \widetilde{L}_8 \sum_{k=1}^{k_{\max}}\sum_{i,j} f_i^{\mu,R}\chi(kr\leqslant|q_i-q_j|\leqslant(k+1)r)\frac{1}{|q_i-q_j|^{\gamma}},
\end{aligned}
$$

where $k_{\max}=[4/3\pi\, R^3/r]+1$, then:

$$
U^{(2)} \leqslant \widetilde{L}_8 \sum_{k=1}^{k_{\max}} \frac{1}{(kr)^{\gamma}} \sum_{\substack{l,m\in\\ \mathbb{Z}^3}}\sum_{\substack{i,j\in\\ \mathbb{N}}} \chi_i(\Delta_l)\chi_j(\Delta_m)
$$

$$
\begin{aligned}
&\quad \times (1+\alpha\sqrt{3})^{\lambda} f(|l-\mu|, R)\ \ \chi(kr-\sqrt{3}\leqslant |l-m|\leqslant (k+1)r+\sqrt{3}) \\
&\leqslant \widetilde{L}_9 \sum_{k=1}^{k_{\max}} \frac{1}{(kr)^{\gamma}} \\
&\quad \times \sum_{\substack{l,m\in\\ \mathbb{Z}^3}} f(|l-\mu|, R)\, n_{\Delta_l} n_{\Delta_m} \chi(kr-\sqrt{3}\leqslant |l-m|\leqslant (k+1)r+\sqrt{3}) \\
&\leqslant \frac{\widetilde{L}_9}{2} \sum_{k=1}^{k_{\max}} \frac{1}{(kr)^{\gamma}} \sum_{\substack{l,m\in\\ \mathbb{Z}^3}} \left(f(|l-\mu|, R) n^2_{\Delta_l} + f(|m-\mu|, R)(1+\alpha|l-m|)^{\lambda} n^2_{\Delta_m} \right) \\
&\quad \times \chi(kr-\sqrt{3}\leqslant |l-m|\leqslant (k+1)r+\sqrt{3}) \\
&\leqslant \widetilde{L}_{10} \sum_{k=1}^{\infty} \frac{k^2 r^3}{(kr)^{\gamma}} \sum_{l\in\mathbb{Z}^3} f(|l-\mu|, R)\, n^2_{\Delta_l} \\
&\quad + \widetilde{L}_{11} \sum_{k=1}^{\infty} \frac{k^2 (1+\alpha((k+1)r+\sqrt{3}))^{\lambda}}{(kr)^{\gamma}} \sum_{m\in\mathbb{Z}^3} f(|m-\mu|, R)\, n^2_{\Delta_m} \\
&\leqslant \widetilde{L}_{12} \sum_{l\in\mathbb{Z}^3} f(|l-\mu|, R)\, n^2_{\Delta_l},
\end{aligned}
\tag{C.7}
$$

where in the last inequality the convergence of the series follows from the bound on γ.

Thus, from Lemma 3.1 we have

$$
Q(X;\mu, R) \leqslant T + U^{(1)} + U^{(2)} \leqslant \widetilde{L}_{13} W(X;\mu, R),
$$

and then the proof of the first inequality of the Corollary follows.

Let us consider the second one.

From the definition of $Q(X;\mu, R)$ and from the superstability of the potential we have

$$
Q(X;\mu, R) \geqslant A \sum_{\substack{k\in\mathbb{Z}^3:\\ |k-\mu|<R}} n^2_{\Delta_k} \geqslant \widetilde{L}_{14} \sum_{\substack{k\in\mathbb{Z}^3:\\ |k-\mu|<R}} f(|k-\mu|, R)\, n^2_{\Delta_k}, \tag{C.8}
$$

and from Corollary 3.1:

$$
\begin{aligned}
W(X;\mu, R) &\leqslant C_4 \sum_{i\in\mathbb{Z}^3:} f(|i-\mu|, R)\, n^2_{\Delta_i} + \sum_{i\in\mathbb{N}} f_i^{\mu, R} \frac{v_i^2}{2} \\
&\leqslant C_4 \sum_{k\geqslant 0} \sum_{\substack{i\in\mathbb{Z}^3:\\ i\in B(\mu,(k+1)R)\setminus B(\mu,kR)}} f(|i-\mu|, R)\, n^2_{\Delta_i}
\end{aligned}
$$

$$+\sum_{k\geqslant 0}\sum_{i\in\mathbb{N}}\chi(kR\leqslant|q_i-\mu|<(k+1)R)\ f_i^{\mu,R}\,\frac{v_i^2}{2}$$

$$\leqslant \widetilde{L}_{15}\sum_{k\geqslant 1}\frac{1}{k^\lambda}\sum_{\substack{i\in\mathbb{Z}^3:\\ i\in B(\mu,(k+1)R)}} n_{\Delta_i}^2$$

$$+\widetilde{L}_{16}\sum_{k\geqslant 0}\frac{1}{(1+k)^\lambda}\,Q(X;\mu,(k+1)R)$$

$$\leqslant \widetilde{L}_{17}\,R^3\sum_{k\geqslant 1}\frac{1}{k^{\lambda-3}}\,\frac{Q(X;\mu,(k+1)R)}{((k+1)R)^3}.$$

Dividing by R^3

$$\frac{W(X;\mu,R)}{R^3}\leqslant \widetilde{L}_{17}\sum_{k\geqslant 1}\frac{1}{k^{\lambda-3}}\,\frac{Q(X;\mu,(k+1)R)}{((k+1)R)^3},$$

from which, taking the supremum over $\mu\in\mathbb{R}^3$ and over $R>\psi_\xi(|\mu|)$

$$\sup_\mu\ \sup_{R>\psi_\xi(|\mu|)}\frac{W(X;\mu,R)}{R^3}\leqslant \widetilde{L}_{18}\,Q_\xi\sum_{k\geqslant 1}\frac{1}{k^{\lambda-3}}\leqslant \widetilde{L}_{19}\,Q_\xi, \tag{C.9}$$

being $\lambda>4$. ▮

APPENDIX D

Proof of Lemma 3.3. (i) Since

$$\frac{1}{(1+\alpha\,\frac{|y|}{nR})^\lambda}\leqslant\frac{n^\lambda}{(1+\alpha\,\frac{|y|}{R})^\lambda}$$

then, from the first two properties of Proposition 2.2, $\exists\,\widetilde{L}_{20}>0$ such that

$$f_i^{\mu,nR}\leqslant \widetilde{L}_{20}\,n^\lambda\,f_i^{\mu,R}.$$

By Corollary 3.1 it follows that

$$\begin{aligned} W(X;\mu,nR) &\leqslant \sum_{i\in\mathbb{N}} f_i^{\mu,nR}\,\frac{v_i^2}{2}+C_4\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,nR)\,n_{\Delta_k}^2\\ &\leqslant \widetilde{L}_{20}\,n^\lambda\,W(X;\mu,R) \end{aligned}$$

$$+C_4\,\widetilde{L}_{20}n^{\lambda}\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\,n^2_{\Delta_k}\leqslant C_8\,n^{\lambda}\,W(X;\mu,R).$$

(ii) From the definition of the weight-function we have $f(x,R_1)<f(x,R_2)$, if $R_1<R_2$.

Using again Corollary 3.1 we get

$$\begin{aligned}W(X;\mu,R)&\leqslant C_4\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,R)\,n^2_{\Delta_k}+\sum_{i\in\mathbb{N}} f_i^{\mu,R}\frac{v_i^2}{2}\\&\leqslant C_4\sum_{k\in\mathbb{Z}^3} f(|k-\mu|,nR)\,n^2_{\Delta_k}+W(X;\mu,nR)\\&\leqslant C_9\,W(X;\mu,nR).\end{aligned}$$

(iii) We use the superstability of the interaction and the bound (C.1):

$$\begin{aligned}W(X;\mu,R)&\geqslant\frac{1}{\widetilde{L}}\,Q(X;\mu,R)\\&\geqslant\frac{1}{2\widetilde{L}}\sum_{i,j}\chi(|q_i-\mu|\leqslant R)\,\chi(|q_j-\mu|\leqslant R)\,\phi_{i,j}\\&\geqslant\frac{\widetilde{L}_{21}}{R^3}\,N^2(X,\mu,R)-B\,\frac{1}{2\widetilde{L}}\,N(X,\mu,R).\end{aligned}$$

Since the interaction energy is positive:

$$N^2(X,\mu,R)\leqslant\widetilde{L}_{22}\,R^3\,(\,N(X,\mu,R)+W(X;\mu,R)\,)\leqslant\widetilde{L}_{23}\,R^3\,W(X;\mu,R).$$

(iv) Let us cover the ball $B(\mu,R)$ by a collection of disjoint cubes $\{\Delta_\alpha\}_{\alpha\in\mathbb{Z}^3}$ of side one. Therefore

$$\sum_{i\neq j}\chi(|q_i-q_j|<\rho)\chi(|q_i-\mu|<R)\chi(|q_j-\mu|<R)\leqslant\sum_{(\alpha,\beta)} n_{\Delta_\alpha}\,n_{\Delta_\beta}+\sum_{\alpha} n^2_{\Delta_\alpha},\tag{D.1}$$

where (α,β) means the sum restricted to all pairs of different cubes at distance not larger than ρ. Thus we have the bound:

$$
\begin{aligned}
&\sum_{i\neq j}\chi(|q_i-q_j|<\rho)\chi(|q_i-\mu|<R)\chi(|q_j-\mu|<R)\\
&\quad\leqslant\sum_{\alpha}n^2_{\Delta_\alpha}+\frac{1}{2}\sum_{(\alpha,\beta)}\left(n^2_{\Delta_\alpha}+n^2_{\Delta_\beta}\right)\leqslant\widetilde{L}_{24}\,\rho^3\sum_{\alpha}n^2_{\Delta_\alpha}\\
&\quad\leqslant\widetilde{L}_{25}\,\rho^3\sum_{i\in\mathbb{Z}^3}f(|i-\mu|,R)\,n^2_{\Delta_i}\leqslant\widetilde{L}_{26}\,\rho^3\,W(X;\mu,R).\quad\blacksquare
\end{aligned}
\tag{D.2}
$$

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